

ON KATO-SOBOLEV TYPE SPACES

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ABSTRACT. We study an increasing family of spaces $\{\mathcal{B}_k^p\}_{1 \leq p \leq \infty}$ by adapting the techniques used in the study of Beurling algebras by Coifman and Meyer [Co-Me]. A weak form Wiener-Levy theorem is proved based on an integral representation formula belonging A. P. Calderón. Also we study the Schatten-von Neumann properties of pseudo-differential operators with symbols in the spaces \mathcal{B}_k^p spaces.

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1. INTRODUCTION

In this paper we study a class of spaces which generalizes the Kato-Sobolev spaces. As we noted in [A2], Kato-Sobolev spaces are particular cases of Wiener amalgam spaces with local component \mathcal{H}^s and global component L^p . Wiener amalgam spaces were introduced by Hans Georg Feichtinger in 1980. Allowing more general weight functions, in this paper we consider as local component the spaces $\mathcal{B}_k = B_{2,k}$ introduced by Lars Hörmander (see [Hö1] vol. 2) and we preserve the global component L^p . Most of the results proved in the case of Kato-Sobolev spaces are preserved except the Wiener-Levy theorem and the spectral analysis for \mathcal{B}_k^p algebras based on this theorem. However, a weak form of this theorem is proved. Also we prove Schatten-von Neumann class properties for pseudo-differential operators with symbols in the spaces \mathcal{B}_k^p . Besides the properties of the spaces $B_{p,k}$, the main techniques we use in the study of these spaces are inspired by techniques used in the study of Beurling algebras by Coifman and Meyer [Co-Me]. Also the proof of the weak form Wiener-Levy theorem is based on an integral representation formula of A. P. Calderón. In Section 2 we recall some properties of the spaces $B_{p,k}$ and we establish the main technical result used in paper by adapting the techniques

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of Coifman and Meyer, used in the study of Beurling algebras. In Section 3 we study an increasing family of spaces $\{\mathcal{B}_k^p\}_{1 \leq p \leq \infty}$. The weak form of Wiener-Lévy theorem for \mathcal{B}_k^p algebras is established in Section 4. The Schatten-von Neumann class properties for pseudo-differential operators with symbols in the spaces \mathcal{B}_k^p are presented in the last section.

2. THE SPACES $\mathcal{B}_k \equiv B_{2,k}$

Let m be an integer ≥ 0 or $m = \infty$. We shall use the following standard notations:

$$\mathcal{BC}^m(\mathbb{R}^n) = \{f \in C^m(\mathbb{R}^n) : f \text{ and its derivatives of order } \leq m \text{ are bounded}\},$$

$$\|f\|_{\mathcal{BC}^l} = \max_{j \leq l} \sup_{x \in \mathbb{R}^n} \|f^{(j)}(x)\| < \infty, \quad l < m + 1.$$

Definition 2.1. Let k be a positive measurable function defined in \mathbb{R}^n . Then:

(a) k will be called a temperate weight function if there are positive constants C and N such that

$$(2.1) \quad k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^n.$$

The set of all such functions k will be denoted by $\mathcal{K}(\mathbb{R}^n)$.

(b) k will be called a weight function of polinomial growth if there are positive constants C and N such that

$$(2.2) \quad k(\xi + \eta) \leq C(1 + |\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^n.$$

For a weight function of polinomial growth k , we shall write

$$M_k(\xi) = \sup_{\eta} \frac{k(\xi + \eta)}{k(\eta)}, \quad \xi \in \mathbb{R}^n.$$

This means that M_k is the smallest function such that

$$k(\xi + \eta) \leq M_k(\xi) k(\eta), \quad \xi, \eta \in \mathbb{R}^n.$$

It is clear that M_k is submultiplicative,

$$M_k(\xi + \eta) \leq M_k(\xi) M_k(\eta), \quad \xi, \eta \in \mathbb{R}^n,$$

and since $M_k(\xi) \leq C(1 + |\xi|)^N$ this implies that

$$1 = M_k(0) \leq M_k(\xi), \quad \xi \in \mathbb{R}^n.$$

In fact, for every positive integer ν we have

$$1 = M_k(0) \leq M_k(\xi)^\nu M_k(-\nu\xi) \leq C M_k(\xi)^\nu (1 + \nu|\xi|)^N$$

and if we take ν^{th} roots in this inequality and let $\nu \rightarrow \infty$, the estimate follows.

Let $k \in \mathcal{K}(\mathbb{R}^n)$. From definition we deduce that

$$(1 + C|\xi|)^{-N} \leq \frac{k(\xi + \eta)}{k(\eta)} \leq (1 + C|\xi|)^N, \quad \xi, \eta \in \mathbb{R}^n.$$

Now if we let $\xi \rightarrow 0$ it follows that k is continuous. If we take $\eta = 0$ we obtain the estimates

$$k(0)(1 + C|\xi|)^{-N} \leq k(\xi) \leq k(0)(1 + C|\xi|)^N, \quad \xi \in \mathbb{R}^n.$$

Submultiplicative property of M_k and $M_k(\xi) \leq (1 + C|\xi|)^N$ implies that $M_k \in \mathcal{K}(\mathbb{R}^n)$. In particular, M_k is continuous for every $k \in \mathcal{K}(\mathbb{R}^n)$.

Lemma 2.2. *Let k be a weight function of polynomial growth. Then for every $\delta > 0$ we can find a function $k_\delta \in \mathcal{K}(\mathbb{R}^n)$ and a constant A_δ such that*

$$1 \leq k_\delta(\xi) / k(\xi) \leq A_\delta, \quad \xi \in \mathbb{R}^n,$$

and $M_{k_\delta} \rightarrow 1$ uniformly on compact subsets of \mathbb{R}^n when $\delta \rightarrow 0$.

Proof. We shall set

$$k_\delta(\xi) = \sup_{\eta} e^{-\delta|\eta|} k(\xi - \eta) = \sup_{\eta} e^{-\delta|\xi - \eta|} k(\eta)$$

Then we have in view of (2.2)

$$k(\xi) \leq k_\delta(\xi) \leq \sup_{\eta} e^{-\delta|\eta|} C(1 + |\eta|)^N k(\xi) = A_\delta k(\xi)$$

where $A_\delta = C \sup_{\eta} e^{-\delta|\eta|} (1 + |\eta|)^N$.

In order to prove that $k_\delta \in \mathcal{K}(\mathbb{R}^n)$ we note that

$$\begin{aligned} k_\delta(\xi + \xi') &= \sup_{\eta} e^{-\delta|\eta|} k(\xi + \xi' - \eta) \leq \sup_{\eta} e^{-\delta|\eta|} C(1 + |\xi'|)^N k(\xi - \eta) \\ &= C(1 + |\xi'|)^N \sup_{\eta} e^{-\delta|\eta|} k(\xi - \eta) = C(1 + |\xi'|)^N k_\delta(\xi) \end{aligned}$$

We also note that

$$\begin{aligned} k_\delta(\xi + \xi') &= \sup_{\eta} e^{-\delta|\xi + \xi' - \eta|} k(\eta) \\ &\leq e^{\delta|\xi'|} \sup_{\eta} e^{-\delta|\xi - \eta|} k(\eta) = e^{\delta|\xi'|} k_\delta(\xi) \end{aligned}$$

Hence

$$\begin{aligned} \frac{k_\delta(\xi + \xi')}{k_\delta(\xi)} &\leq e^{\delta|\xi'|} \text{car}_{\mathbf{B}}(\xi') + C(1 + |\xi'|)^N \text{car}_{\mathbf{CB}}(\xi') \\ &\leq (1 + C_\delta |\xi'|)^N \end{aligned}$$

where $\mathbf{B} = \{\xi : |\xi| \leq 1\}$ and $C_\delta = \max \left\{ \frac{\delta}{N} e^{\frac{\delta}{N}}, 2 \cdot C^{\frac{1}{N}} - 1 \right\} \geq 1$. Here we used the elementary estimates

$$0 \leq t \leq 1 \Rightarrow e^{\frac{\delta}{N}t} - 1 = \frac{\delta}{N} \int_0^t e^{\frac{\delta}{N}\tau} d\tau \leq \frac{\delta}{N} e^{\frac{\delta}{N}t} \leq C_\delta t \Rightarrow e^{\delta t} \leq (1 + C_\delta t)^N,$$

and

$$\begin{aligned} C_\delta \geq 1 &\Rightarrow t \rightarrow \frac{1 + C_\delta t}{1 + t} \nearrow \Rightarrow t \rightarrow \left(\frac{1 + C_\delta t}{1 + t} \right)^N \nearrow \\ &\Rightarrow \inf_{t \geq 1} \left(\frac{1 + C_\delta t}{1 + t} \right)^N = \left(\frac{1 + C_\delta}{2} \right)^N \geq \left(\frac{1 + 2 \cdot C^{\frac{1}{N}} - 1}{2} \right)^N = C. \end{aligned}$$

Since

$$1 \leq M_{k_\delta}(\xi') = \sup_{\xi} \frac{k_\delta(\xi + \xi')}{k_\delta(\xi)} \leq e^{\delta|\xi'|}, \quad \xi' \in \mathbb{R}^n,$$

$M_{k_\delta} \rightarrow 1$ uniformly on compact subsets of \mathbb{R}^n when $\delta \rightarrow 0$. The proof is complete. \square

Definition 2.3. If $k \in \mathcal{K}(\mathbb{R}^n)$ and $1 \leq p \leq \infty$, we denote by $B_{p,k}(\mathbb{R}^n)$ the set of all distributions $u \in \mathcal{S}'$ such that \widehat{u} is a function and $k\widehat{u} \in L^p$. For $u \in B_{p,k}(\mathbb{R}^n)$ we define

$$\|u\|_{p,k} = \|k\widehat{u}\|_p < \infty$$

Lemma 2.4. Let $k \in \mathcal{K}(\mathbb{R}^n)$ and C, N the positive constants that define k and $1 \leq p \leq \infty$. Let $m_k = [N + \frac{n+1}{2}] + 1$ and $l_k = [N] + n + 2$.

(a) If $\chi \in H^{N+\frac{n+1}{2}}$, then for every $u \in B_{p,k}(\mathbb{R}^n)$ we have $\chi u \in B_{p,k}(\mathbb{R}^n)$ and

$$\begin{aligned} \|\chi u\|_{p,k} &\leq (2\pi)^{-n} \|M_k \widehat{\chi}\|_1 \|u\|_{p,k} = C(k, n, \chi) \|u\|_{p,k} \\ &\leq C(C, N, n) \|\chi\|_{H^{N+\frac{n+1}{2}}} \|u\|_{p,k}, \end{aligned}$$

where

$$\begin{aligned} C(k, n, \chi) &= (2\pi)^{-n} \|M_k \widehat{\chi}\|_1 \leq (2\pi)^{-n} \left(\int M_k(\eta) |\widehat{\chi}(\eta)| d\eta \right) \\ &\leq (2\pi)^{-n} (\max\{1, C\})^N 2^{N/2} \left\| \langle \cdot \rangle^{-n-1} \right\|_{L^1} \|\chi\|_{H^{N+\frac{n+1}{2}}} \\ &= C(C, N, n) \|\chi\|_{H^{N+\frac{n+1}{2}}}. \end{aligned}$$

Here $H^m(\mathbb{R}^n)$ is the usual Sobolev space, $m \in \mathbb{R}$. If $\chi \in H^{m_k}$, then

$$\|\chi u\|_{p,k} \leq C(C, N, n) \left(\sum_{|\alpha| \leq m_k} \|\partial^\alpha \chi\|_{L^2} \right) \|u\|_{p,k}.$$

(b) If $\chi \in \mathcal{C}^{l_k}(\mathbb{R}^n)$ is \mathbb{Z}^n -periodic, then for every $u \in B_{p,k}(\mathbb{R}^n)$ we have $\chi u \in B_{p,k}(\mathbb{R}^n)$ and

$$\|\chi u\|_{p,k} \leq Cst(C, N, n) \|\chi\|_{\mathcal{BC}^{l_k}(\mathbb{R}^n)} \|u\|_{p,k}.$$

(c) If $1/k \in L^{p'}(\mathbb{R}^n)$, $1/p + 1/p' = 1$, then $B_{p,k}(\mathbb{R}^n) \subset \mathcal{F}^{-1}L^1(\mathbb{R}^n) \subset \mathcal{C}_\infty(\mathbb{R}^n)$.

Proof. (a) Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{N+\frac{n+1}{2}}(\mathbb{R}^n)$, we can assume that $\chi \in \mathcal{S}(\mathbb{R}^n)$. We know from Theorem 10.1.15 in Hörmander [Hö1] vol. 2 that

$$\|\chi u\|_{p,k} \leq (2\pi)^{-n} \|M_k \widehat{\chi}\|_1 \|u\|_{p,k}.$$

Since

$$M_k(\xi) \leq (1 + C|\xi|)^N \leq 2^{N/2} (\max\{1, C\})^N \langle \xi \rangle^N,$$

Schwarz inequality gives the estimate of $C(k, n, \chi)$

$$\begin{aligned} C(k, n, \chi) &= (2\pi)^{-n} \|M_k \widehat{\chi}\|_1 \leq (2\pi)^{-n} \left(\int M_k(\eta) |\widehat{\chi}(\eta)| d\eta \right) \\ &\leq (2\pi)^{-n} (\max\{1, C\})^N 2^{N/2} \left(\int \langle \eta \rangle^N |\widehat{\chi}(\eta)| d\eta \right) \\ &\leq (2\pi)^{-n} (\max\{1, C\})^N 2^{N/2} \left\| \langle \cdot \rangle^{-n-1} \right\|_{L^1} \|\chi\|_{H^{N+\frac{n+1}{2}}} \\ &= C(C, N, n) \|\chi\|_{H^{N+\frac{n+1}{2}}}, \end{aligned}$$

with $\|\chi\|_{H^{N+\frac{n+1}{2}}} \leq Cst \left(\sum_{|\alpha| \leq m_k} \|\partial^\alpha \chi\|_{L^2} \right)$ when $\chi \in H^{m_k}(\mathbb{R}^n)$.

(b) We shall use some results from [Hö1] vol. 1, pp 177-179, concerning periodic distributions. If $\chi \in \mathcal{C}^{l_k}(\mathbb{R}^n)$ is \mathbb{Z}^n -periodic, then

$$\chi = \sum_{\gamma \in \mathbb{Z}^n} e^{2\pi i \langle \cdot, \gamma \rangle} c_\gamma,$$

with Fourier coefficients

$$c_\gamma = \int_{\mathbf{I}} \chi(x) e^{-2\pi i \langle x, \gamma \rangle} dx, \quad \mathbf{I} = [0, 1]^n, \quad \gamma \in \mathbb{Z}^n,$$

satisfying

$$|c_\gamma| \leq Cst \|\chi\|_{\mathcal{BC}^{l_k}(\mathbb{R}^n)} \langle 2\pi\gamma \rangle^{-l_k}, \quad \gamma \in \mathbb{Z}^n.$$

Since $\widehat{e^{i\langle \cdot, \eta \rangle} u} = \widehat{u}(\cdot - \eta)$, multiplying by $k(\xi)$ and noting the inequality $k(\xi) \leq M_k(\eta) k(\xi - \eta)$, we obtain

$$\begin{aligned} \left| k(\xi) \widehat{e^{i\langle \cdot, \eta \rangle} u}(\xi) \right| &\leq M_k(\eta) |k(\xi - \eta) \widehat{u}(\xi - \eta)| \\ &\leq 2^{N/2} (\max\{1, C\})^N \langle \eta \rangle^N |k(\xi - \eta) \widehat{u}(\xi - \eta)|, \end{aligned}$$

and

$$\left\| e^{i\langle \cdot, \eta \rangle} u \right\|_{p,k} \leq 2^{N/2} (\max\{1, C\})^N \langle \eta \rangle^N \|u\|_{p,k}.$$

It follows that

$$\begin{aligned} \|\chi u\|_{p,k} &\leq Cst \cdot 2^{N/2} (\max\{1, C\})^N \|\chi\|_{\mathcal{BC}^{l_k}(\mathbb{R}^n)} \left(\sum_{\gamma \in \mathbb{Z}^n} \langle 2\pi\gamma \rangle^{-l_k} \langle 2\pi\gamma \rangle^N \right) \|u\|_{p,k} \\ &\leq Cst \cdot 2^{N/2} (\max\{1, C\})^N \left(\sum_{\gamma \in \mathbb{Z}^n} \langle 2\pi\gamma \rangle^{-n-1} \right) \|\chi\|_{\mathcal{BC}^{l_k}(\mathbb{R}^n)} \|u\|_{p,k}. \end{aligned}$$

since $l_k = [N] + n + 2$.

(c) Let $u \in B_{p,k}(\mathbb{R}^n)$. If $1/k \in L^{p'}$, then $\widehat{u} \in L^1(\mathbb{R}^n)$ since $1/k \in L^{p'}(\mathbb{R}^n)$, $k\widehat{u} \in L^p(\mathbb{R}^n)$ and $1/p + 1/p' = 1$. Now the Riemann-Lebesgue lemma implies the result. For $x \in \mathbb{R}^n$ we have

$$|u(x)| \leq (2\pi)^{-n} \|\widehat{u}\|_{L^1} \leq (2\pi)^{-n} \|k^{-1}\|_{L^{p'}} \|k\widehat{u}\|_{L^p} = (2\pi)^{-n} \|k^{-1}\|_{L^{p'}} \|u\|_{p,k}.$$

□

Lemma 2.5. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\theta \in [0, 2\pi]^n$. If*

$$\varphi_\theta = \sum_{\gamma \in \mathbb{Z}^n} e^{i\langle \gamma, \theta \rangle} \varphi(\cdot - \gamma) = \sum_{\gamma \in \mathbb{Z}^n} e^{i\langle \gamma, \theta \rangle} \tau_\gamma \varphi,$$

then

$$\widehat{\varphi}_\theta = \nu_\theta = (2\pi)^n \sum_{\gamma \in \mathbb{Z}^n} \widehat{\varphi}(2\pi\gamma + \theta) \delta_{2\pi\gamma + \theta}.$$

Proof. We have

$$\varphi_\theta = \sum_{\gamma \in \mathbb{Z}^n} e^{i\langle \gamma, \theta \rangle} \varphi(\cdot - \gamma) = \sum_{\gamma \in \mathbb{Z}^n} e^{i\langle \gamma, \theta \rangle} \delta_\gamma * \varphi = \varphi * \left(e^{i\langle \cdot, \theta \rangle} S \right),$$

where $S = \sum_{\gamma \in \mathbb{Z}^n} \delta_\gamma$. We apply Poisson's summation formula, $\mathcal{F}\left(\sum_{\gamma \in \mathbb{Z}^n} \delta_\gamma\right) = (2\pi)^n \sum_{\gamma \in \mathbb{Z}^n} \delta_{2\pi\gamma}$, to obtain

$$\begin{aligned} \widehat{\varphi}_\theta &= \widehat{\varphi} \cdot \left(\widehat{e^{i\langle \cdot, \theta \rangle} S} \right) = \widehat{\varphi} \cdot \tau_\theta \widehat{S} = (2\pi)^n \widehat{\varphi} \sum_{\gamma \in \mathbb{Z}^n} \delta_{2\pi\gamma + \theta} \\ &= (2\pi)^n \sum_{\gamma \in \mathbb{Z}^n} \widehat{\varphi}(2\pi\gamma + \theta) \delta_{2\pi\gamma + \theta}. \end{aligned}$$

□

Above and in the rest of the paper for any $x \in \mathbb{R}^n$ and for any distribution u on \mathbb{R}^n , by $\tau_x u$ we shall denote the translation by x of u , i.e. $\tau_x u = u(\cdot - x) = \delta_x * u$.

Notation 1. For k in $\mathcal{K}(\mathbb{R}^n)$ we denote by $\mathcal{B}_k(\mathbb{R}^n)$ the Hilbert space $B_{2,k}(\mathbb{R}^n)$. We shall use $\|\cdot\|_{\mathcal{B}_k}$ for the norm $\|\cdot\|_{2,k}$.

As we already said the techniques of Coifman and Meyer, used in the study of Beurling algebras A_ω and B_ω (see [Co-Me] pp 7-10), can be adapted to the case spaces $\mathcal{B}_k(\mathbb{R}^n) = B_{2,k}(\mathbb{R}^n)$. An example is the following result.

Lemma 2.6. Let $k \in \mathcal{K}(\mathbb{R}^n)$. Let $\{u_\gamma\}_{\gamma \in \mathbb{Z}^n}$ be a family of elements from $\mathcal{B}_k(\mathbb{R}^n) \cap \mathcal{D}'_K(\mathbb{R}^n)$, where $K \subset \mathbb{R}^n$ is a compact subset such that $(K - K) \cap \mathbb{Z}^n = \{0\}$. Put

$$u = \sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma u_\gamma = \sum_{\gamma \in \mathbb{Z}^n} u_\gamma(\cdot - \gamma) = \sum_{\gamma \in \mathbb{Z}^n} \delta_\gamma * u_\gamma \in \mathcal{D}'(\mathbb{R}^n).$$

Then the following statements are equivalent:

- (a) $u \in \mathcal{B}_k(\mathbb{R}^n)$.
- (b) $\sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2 < \infty$.

Moreover, there is $C \geq 1$, which does not depend on the family $\{u_\gamma\}_{\gamma \in \mathbb{Z}^n}$, such that

$$(2.3) \quad C^{-1} \|u\|_{\mathcal{B}_k} \leq \left(\sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2 \right)^{1/2} \leq C \|u\|_{\mathcal{B}_k}.$$

Proof. Let us choose $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on K and $\text{supp } \varphi = K'$ satisfies the condition $(K' - K') \cap \mathbb{Z}^n = \{0\}$. For $\theta \in [0, 2\pi]^n$ we set

$$\begin{aligned} \varphi_\theta &= \sum_{\gamma \in \mathbb{Z}^n} e^{i\langle \gamma, \theta \rangle} \tau_\gamma \varphi = \sum_{\gamma \in \mathbb{Z}^n} e^{i\langle \gamma, \theta \rangle} \delta_\gamma * \varphi, \\ u_\theta &= \sum_{\gamma \in \mathbb{Z}^n} e^{i\langle \gamma, \theta \rangle} \tau_\gamma u_\gamma = \sum_{\gamma \in \mathbb{Z}^n} e^{i\langle \gamma, \theta \rangle} \delta_\gamma * u_\gamma. \end{aligned}$$

Since $(K' - K') \cap \mathbb{Z}^n = \{0\}$ we have

$$u_\theta = \varphi_\theta u, \quad u = \varphi_\theta u_{-\theta}.$$

Step 1. Suppose first that the family $\{u_\gamma\}_{\gamma \in \mathbb{Z}^n}$ has only a finite number of non-zero terms and we shall prove in this case the estimate (2.3). Since $u_\theta, u \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ it follows that

$$\hat{u}_\theta = (2\pi)^{-n} \nu_\theta * \hat{u}, \quad \hat{u} = (2\pi)^{-n} \nu_\theta * \hat{u}_{-\theta},$$

where $\nu_\theta = \hat{\varphi}_\theta = (2\pi)^n \sum_{\gamma \in \mathbb{Z}^n} \hat{\varphi}(2\pi\gamma + \theta) \delta_{2\pi\gamma + \theta}$ is a measure of rapid decay at ∞ . Since $\hat{u}_\theta, \hat{u} \in \mathcal{C}_{pol}^\infty(\mathbb{R}^n)$ we get the pointwise equalities

$$\begin{aligned} \hat{u}_\theta(\xi) &= \sum_{\gamma \in \mathbb{Z}^n} \hat{\varphi}(2\pi\gamma + \theta) \hat{u}(\xi - 2\pi\gamma - \theta), \\ \hat{u}(\xi) &= \sum_{\gamma \in \mathbb{Z}^n} \hat{\varphi}(2\pi\gamma + \theta) \hat{u}_{-\theta}(\xi - 2\pi\gamma - \theta). \end{aligned}$$

Multiplying by $k(\xi)$ and using the inequality $k(\xi) \leq M_k(2\pi\gamma + \theta)k(\xi - 2\pi\gamma - \theta)$ we obtain

$$k(\xi) |\widehat{u}_\theta(\xi)| \leq \sum_{\gamma \in \mathbb{Z}^n} M_k(2\pi\gamma + \theta) |\widehat{\varphi}(2\pi\gamma + \theta)| \cdot k(\xi - 2\pi\gamma - \theta) |\widehat{u}(\xi - 2\pi\gamma - \theta)|,$$

and

$$k(\xi) \widehat{u}(\xi) \leq \sum_{\gamma \in \mathbb{Z}^n} M_k(2\pi\gamma + \theta) |\widehat{\varphi}(2\pi\gamma + \theta)| \cdot k(\xi - 2\pi\gamma - \theta) |\widehat{u}_{-\theta}(\xi - 2\pi\gamma - \theta)|.$$

It follows that

$$\begin{aligned} \|u_\theta\|_{\mathcal{B}_k} &= \|k\widehat{u}_\theta\|_{L^2} \\ &\leq \left(\sum_{\gamma \in \mathbb{Z}^n} M_k(2\pi\gamma + \theta) |\widehat{\varphi}(2\pi\gamma + \theta)| \right) \|k\widehat{u}\|_{L^2} \\ &= \left(\sum_{\gamma \in \mathbb{Z}^n} M_k(2\pi\gamma + \theta) |\widehat{\varphi}(2\pi\gamma + \theta)| \right) \|u\|_{\mathcal{B}_k} \\ &= C_{k,\varphi} \|u\|_{\mathcal{B}_k} \end{aligned}$$

and

$$\begin{aligned} \|u\|_{\mathcal{B}_k} &\leq \left(\sum_{\gamma \in \mathbb{Z}^n} M_k(2\pi\gamma + \theta) |\widehat{\varphi}(2\pi\gamma + \theta)| \right) \|u_{-\theta}\|_{\mathcal{B}_k} \\ &= C_{k,\varphi} \|u_{-\theta}\|_{\mathcal{B}_k}, \end{aligned}$$

where $C_{k,\varphi} = \sum_{\gamma \in \mathbb{Z}^n} M_k(2\pi\gamma + \theta) |\widehat{\varphi}(2\pi\gamma + \theta)| < \infty$ because

$$M_k(\xi) \leq (1 + C|\xi|)^N \leq 2^{N/2} (\max\{1, C\})^N \langle \xi \rangle^N.$$

The above estimates can be rewritten as

$$\begin{aligned} \int |k(\xi) \widehat{u}_\theta(\xi)|^2 d\xi &\leq C_{k,\varphi}^2 \|u\|_{\mathcal{B}_k}^2, \\ \|u\|_{\mathcal{B}_k}^2 &\leq C_{k,\varphi}^2 \int |k(\xi) \widehat{u}_{-\theta}(\xi)|^2 d\xi. \end{aligned}$$

On the other hand, the equality $u_\theta = \sum_{\gamma \in \mathbb{Z}^n} \mathbf{e}^{i\langle \gamma, \theta \rangle} \tau_\gamma u_\gamma$ implies

$$\widehat{u}_\theta(\xi) = \sum_{\gamma \in \mathbb{Z}^n} \mathbf{e}^{i\langle \gamma, \theta - \xi \rangle} \widehat{u}_\gamma(\xi)$$

with finite sum. The functions $\theta \rightarrow \widehat{u}_{\pm\theta}(\xi)$ are in $L^2([0, 2\pi]^n)$ and

$$(2\pi)^{-n} \int_{[0, 2\pi]^n} |\widehat{u}_{\pm\theta}(\xi)|^2 d\theta = \sum_{\gamma \in \mathbb{Z}^n} |\widehat{u}_\gamma(\xi)|^2.$$

Integrating with respect θ the above inequalities we get that

$$\begin{aligned} \sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2 &\leq C_{k,\varphi}^2 \|u\|_{\mathcal{B}_k}^2, \\ \|u\|_{\mathcal{B}_k}^2 &\leq C_{k,\varphi}^2 \sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2. \end{aligned}$$

Step 2. The general case is obtained by approximation.

Suppose that $u \in \mathcal{B}_k(\mathbb{R}^n)$. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ on $B(0, 1)$. Then $\psi^\varepsilon u \rightarrow u$ in $\mathcal{B}_k(\mathbb{R}^n)$ where $\psi^\varepsilon(x) = \psi(\varepsilon x)$, $0 < \varepsilon \leq 1$, $x \in \mathbb{R}^n$. Also we have

$$\|\psi^\varepsilon u\|_{\mathcal{B}_k} \leq C(k, \psi) \|u\|_{\mathcal{B}_k}, \quad 0 < \varepsilon \leq 1,$$

where

$$\begin{aligned} C(k, \psi) &= (2\pi)^{-n} \sup_{0 < \varepsilon \leq 1} \left(\int M_k(\eta) \varepsilon^{-n} |\widehat{\psi}(\eta/\varepsilon)| d\eta \right) \\ &\leq (2\pi)^{-n} 2^{N/2} (\max\{1, C\})^N \sup_{0 < \varepsilon \leq 1} \left(\int \langle \varepsilon \eta \rangle^N |\widehat{\psi}(\eta)| d\eta \right) \\ &\leq (2\pi)^{-n} 2^{N/2} (\max\{1, C\})^N \left(\int \langle \eta \rangle^N |\widehat{\psi}(\eta)| d\eta \right). \end{aligned}$$

Let $m \in \mathbb{N}$, $m \geq 1$. Then there is ε_m such that for any $\varepsilon \in (0, \varepsilon_m]$ we have

$$\psi^\varepsilon u = \sum_{|\gamma| \leq m} \tau_\gamma u_\gamma + \sum_{finite} \tau_\gamma ((\tau_{-\gamma} \psi^\varepsilon) u_\gamma).$$

By the first part we get that

$$\sum_{|\gamma| \leq m} \|u_\gamma\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \|\psi^\varepsilon u\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 C(k, \psi)^2 \|u\|_{\mathcal{B}_k}^2.$$

Since m is arbitrary, it follows that $\sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2 < \infty$. Further from

$$\sum_{|\gamma| \leq m} \|u_\gamma\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \|\psi^\varepsilon u\|_{\mathcal{B}_k}^2, \quad 0 < \varepsilon \leq \varepsilon_m,$$

we obtain that

$$\sum_{|\gamma| \leq m} \|u_\gamma\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \|u\|_{\mathcal{B}_k}^2, \quad \forall m \in \mathbb{N}.$$

Hence

$$\sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \|u\|_{\mathcal{B}_k}^2.$$

Now suppose that $\sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2 < \infty$. For $m \in \mathbb{N}$, $m \geq 1$ we put $u(m) = \sum_{|\gamma| \leq m} \tau_\gamma u_\gamma$. Then

$$\|u(m+p) - u(m)\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \sum_{m \leq |\gamma| \leq m+p} \|u_\gamma\|_{\mathcal{B}_k}^2$$

It follows that $\{u(m)\}_{m \geq 1}$ is a Cauchy sequence in $\mathcal{B}_k(\mathbb{R}^n)$. Let $v \in \mathcal{B}_k(\mathbb{R}^n)$ be such that $u(m) \rightarrow v$ in $\mathcal{B}_k(\mathbb{R}^n)$. Since $u(m) \rightarrow u$ in $\mathcal{D}'(\mathbb{R}^n)$, it follows that $u = v$. Hence $u(m) \rightarrow u$ in $\mathcal{B}_k(\mathbb{R}^n)$. Since we have

$$\|u(m)\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \sum_{|\gamma| \leq m} \|u_\gamma\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2, \quad \forall m \in \mathbb{N}.$$

we obtain that

$$\|u\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2.$$

□

To use the previous result we need a convenient partition of unity. Let $m \in \mathbb{N}$ and $\{x_1, \dots, x_m\} \subset \mathbb{R}^n$ be such that

$$[0, 1]^n \subset \left(x_1 + \left[\frac{1}{3}, \frac{2}{3}\right]^n\right) \cup \dots \cup \left(x_m + \left[\frac{1}{3}, \frac{2}{3}\right]^n\right)$$

Let $\tilde{h} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\tilde{h} \geq 0$, be such that $\tilde{h} = 1$ on $[\frac{1}{3}, \frac{2}{3}]^n$ and $\text{supp } \tilde{h} \subset [\frac{1}{4}, \frac{3}{4}]^n$. Then

- (a) $\tilde{H} = \sum_{i=1}^m \sum_{\gamma \in \mathbb{Z}^n} \tau_{\gamma+x_i} \tilde{h} \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is \mathbb{Z}^n -periodic and $\tilde{H} \geq 1$.
- (b) $h_i = \frac{\tau_{x_i} \tilde{h}}{\tilde{H}} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h_i \geq 0$, $\text{supp } h_i \subset x_i + [\frac{1}{4}, \frac{3}{4}]^n = K_i$, $(K_i - K_i) \cap \mathbb{Z}^n = \{0\}$, $i = 1, \dots, m$.
- (c) $\chi_i = \sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma h_i \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is \mathbb{Z}^n -periodic, $i = 1, \dots, m$ and $\sum_{i=1}^m \chi_i = 1$.
- (d) $h = \sum_{i=1}^m h_i \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h \geq 0$, $\sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma h = 1$.

A first consequence of previous results is the next proposition.

Proposition 2.7. *Let $k \in \mathcal{K}(\mathbb{R}^n)$ and C, N the positive constants that define k . Let $m_k = \lceil N + \frac{n+1}{2} \rceil + 1$. Then*

$$\mathcal{BC}^{m_k}(\mathbb{R}^n) \cdot \mathcal{B}_k(\mathbb{R}^n) \subset \mathcal{B}_k(\mathbb{R}^n).$$

Proof. Let $u \in \mathcal{B}_k(\mathbb{R}^n)$. We use the partition of unity constructed above to obtain a decomposition of u satisfying the conditions of Lemma 2.6. Using Lemma 2.4 (c), it follows that $\chi_i u \in \mathcal{B}_k(\mathbb{R}^n)$, $i = 1, \dots, m$. We have

$$u = \sum_{i=1}^m \chi_i u$$

with $\chi_i u \in \mathcal{B}_k(\mathbb{R}^n)$,

$$\begin{aligned} \chi_i u &= \sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma (h_i \tau_{-\gamma} u), \quad h_i \tau_{-\gamma} u \in \mathcal{B}_k(\mathbb{R}^n) \cap \mathcal{D}'_{K_i}(\mathbb{R}^n), \\ (K_i - K_i) \cap \mathbb{Z}^n &= \{0\}, \quad i = 1, \dots, m. \end{aligned}$$

So we can assume that $u \in \mathcal{B}_k(\mathbb{R}^n)$ is of the form described in Lemma 2.6.

Let $\psi \in \mathcal{BC}^{m_k}(\mathbb{R}^n)$. Then

$$\psi u = \sum_{\gamma \in \mathbb{Z}^n} \psi \tau_\gamma u_\gamma = \sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma (\psi_\gamma u_\gamma)$$

with $\psi_\gamma = \varphi(\tau_{-\gamma} \psi)$, where $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is the function considered in the proof of Lemma 2.6. We apply Lemma 2.6 and Lemma 2.4 (a) to obtain

$$\|\psi u\|_{\mathcal{B}_k}^2 \leq C_{k,\varphi}^2 \sum_{\gamma \in \mathbb{Z}^n} \|\psi_\gamma u_\gamma\|_{\mathcal{B}_k}^2$$

and

$$\begin{aligned} \|\psi_\gamma u_\gamma\|_{\mathcal{B}_k} &\leq Cst \left(\sum_{|\alpha| \leq m_k} \|\partial^\alpha (\varphi(\tau_{-\gamma} \psi))\|_{L^2} \right) \|u_\gamma\|_{\mathcal{B}_k} \\ &\leq Cst \|\varphi\|_{H^{m_k}} \|\psi\|_{\mathcal{BC}^{m_k}} \|u_\gamma\|_{\mathcal{B}_k}, \quad \gamma \in \mathbb{Z}^n. \end{aligned}$$

Hence another application of Lemma 2.6 gives

$$\begin{aligned} \|\psi u\|_{\mathcal{B}_k}^2 &\leq Cst \|\varphi\|_{H^{m_k}}^2 \|\psi\|_{\mathcal{BC}^{m_k}}^2 \sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2 \\ &\leq Cst \|\varphi\|_{H^{m_k}}^2 \|\psi\|_{\mathcal{BC}^{m_k}}^2 \|u\|_{\mathcal{B}_k}^2. \end{aligned}$$

□

Corollary 2.8. *Let $k \in \mathcal{K}(\mathbb{R}^n)$. Then*

$$\mathcal{BC}^\infty(\mathbb{R}^n) \cdot \mathcal{B}_k(\mathbb{R}^n) \subset \mathcal{B}_k(\mathbb{R}^n).$$

3. THE SPACES \mathcal{B}_k^p

We begin by proving some results that will be useful later. Let $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$). Then the maps

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\ni (x, y) \xrightarrow{f} \varphi(x) \psi(x - y) = (\varphi \tau_y \psi)(x) \in \mathbb{C}, \\ \mathbb{R}^n \times \mathbb{R}^n &\ni (x, y) \xrightarrow{g} \varphi(y) \psi(x - y) = \varphi(y) (\tau_y \psi)(x) \in \mathbb{C}, \end{aligned}$$

are in $\mathcal{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (respectively in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$). To see this we note that

$$f = (\varphi \otimes \psi) \circ T, \quad g = (\varphi \otimes \psi) \circ S$$

where

$$\begin{aligned} T : \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \times \mathbb{R}^n, \quad T(x, y) = (x, x - y), \quad T \equiv \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}, \\ S : \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \times \mathbb{R}^n, \quad S(x, y) = (y, x - y), \quad S \equiv \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}. \end{aligned}$$

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$). Then using Fubini theorem for distributions we get

$$\begin{aligned} \langle u \otimes 1, f \rangle &= \langle (u \otimes 1)(x, y), \varphi(x) \psi(x - y) \rangle \\ &= \langle u(x), \langle 1(y), \varphi(x) \psi(x - y) \rangle \rangle \\ &= \langle u(x), \varphi(x) \langle 1(y), \psi(x - y) \rangle \rangle \\ &= \left\langle u(x), \varphi(x) \int \psi(x - y) \, dy \right\rangle \\ &= \left(\int \psi \right) \langle u, \varphi \rangle \end{aligned}$$

and

$$\begin{aligned} \langle u \otimes 1, f \rangle &= \langle 1(y), \langle u(x), \varphi(x) \psi(x - y) \rangle \rangle \\ &= \int \langle u, \varphi \tau_y \psi \rangle \, dy. \end{aligned}$$

It follows that

$$\left(\int \psi \right) \langle u, \varphi \rangle = \int \langle u, \varphi \tau_y \psi \rangle \, dy$$

valid for

- (i) $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$;
- (ii) $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

We also have

$$\begin{aligned}
\langle u \otimes 1, g \rangle &= \langle (u \otimes 1)(x, y), \varphi(y) \psi(x - y) \rangle \\
&= \langle u(x), \langle 1(y), \varphi(y) \psi(x - y) \rangle \rangle \\
&= \langle u(x), (\varphi * \psi)(x) \rangle \\
&= \langle u, \varphi * \psi \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle u \otimes 1, g \rangle &= \langle 1(y), \langle u(x), \varphi(y) \psi(x - y) \rangle \rangle \\
&= \int \varphi(y) \langle u, \tau_y \psi \rangle dy.
\end{aligned}$$

Hence

$$\langle u, \varphi * \psi \rangle = \int \varphi(y) \langle u, \tau_y \psi \rangle dy$$

true for

- (i) $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$;
- (ii) $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 3.1. *Let $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$) and $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$). Then*

$$(3.1) \quad \left(\int \psi \right) \langle u, \varphi \rangle = \int \langle u, \varphi \tau_y \psi \rangle dy$$

$$(3.2) \quad \langle u, \varphi * \psi \rangle = \int \varphi(y) \langle u, \tau_y \psi \rangle dy$$

If $\varepsilon_1, \dots, \varepsilon_n$ is a basis in \mathbb{R}^n , we say that $\Gamma = \bigoplus_{j=1}^n \mathbb{Z} \varepsilon_j$ is a lattice.

Let $\Gamma \subset \mathbb{R}^n$ be a lattice. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then $\sum_{\gamma \in \Gamma} \tau_\gamma \psi = \sum_{\gamma \in \Gamma} \psi(\cdot - \gamma)$ is uniformly convergent on compact subsets of \mathbb{R}^n . Since $\partial^\alpha \psi \in \mathcal{S}(\mathbb{R}^n)$, it follows that there is $\Psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\Psi = \sum_{\gamma \in \Gamma} \tau_\gamma \psi = \sum_{\gamma \in \Gamma} \psi(\cdot - \gamma) \quad \text{in } \mathcal{C}^\infty(\mathbb{R}^n).$$

Moreover we have $\tau_\gamma \Psi = \Psi(\cdot - \gamma) = \Psi$ for any $\gamma \in \Gamma$. From here we obtain that $\Psi \in \mathcal{BC}^\infty(\mathbb{R}^n)$. If $\Psi(y) \neq 0$ for any $y \in \mathbb{R}^n$, then $\frac{1}{\Psi} \in \mathcal{BC}^\infty(\mathbb{R}^n)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\varphi \Psi = \sum_{\gamma \in \Gamma} \varphi(\tau_\gamma \psi)$$

with the series convergent in $\mathcal{S}(\mathbb{R}^n)$. Indeed we have

$$\begin{aligned}
&\sum_{\gamma \in \Gamma} \langle x \rangle^k |\partial^\alpha \varphi(x) \partial^\beta \psi(x - \gamma)| \\
&\leq \sup_y \langle y \rangle^{n+1} |\partial^\beta \psi(y)| \sum_{\gamma \in \Gamma} \langle x \rangle^k |\partial^\alpha \varphi(x) \langle x - \gamma \rangle^{-n-1}| \\
&\leq 2^{\frac{n+1}{2}} \sup_y \langle y \rangle^{n+1} |\partial^\beta \psi(y)| \sup_z \langle z \rangle^{k+n+1} |\partial^\alpha \varphi(z)| \sum_{\gamma \in \Gamma} \langle \gamma \rangle^{-n-1}.
\end{aligned}$$

This estimate proves the convergence of the series in $\mathcal{S}(\mathbb{R}^n)$. Let χ be the sum of the series $\sum_{\gamma \in \Gamma} \varphi(\tau_\gamma \psi)$ in $\mathcal{S}(\mathbb{R}^n)$. Then for any $y \in \mathbb{R}^n$ we have

$$\begin{aligned} \chi(y) &= \langle \delta_y, \chi \rangle = \left\langle \delta_y, \sum_{\gamma \in \Gamma} \varphi(\tau_\gamma \psi) \right\rangle \\ &= \sum_{\gamma \in \Gamma} \langle \delta_y, \varphi(\tau_\gamma \psi) \rangle = \sum_{\gamma \in \Gamma} \varphi(y) \psi(y - \gamma) \\ &= \varphi(y) \Psi(y). \end{aligned}$$

So $\varphi\Psi = \sum_{\gamma \in \Gamma} \varphi(\tau_\gamma \psi)$ in $\mathcal{S}(\mathbb{R}^n)$.

If $\psi, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is replaced by $\mathcal{C}_0^\infty(\mathbb{R}^n)$, then the previous observations are trivial.

Lemma 3.2. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\psi, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$). Then $\Psi = \sum_{\gamma \in \Gamma} \tau_\gamma \psi \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is Γ -periodic and*

$$(3.3) \quad \langle u, \Psi\varphi \rangle = \sum_{\gamma \in \Gamma} \langle u, (\tau_\gamma \psi) \varphi \rangle.$$

Lemma 3.3. (a) *Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $\widehat{\chi}u \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}_{pol}^\infty(\mathbb{R}^n)$. In fact we have*

$$\widehat{\chi}u(\xi) = \left\langle e^{-i\langle \cdot, \xi \rangle} u, \chi \right\rangle = \left\langle u, e^{-i\langle \cdot, \xi \rangle} \chi \right\rangle, \quad \xi \in \mathbb{R}^n.$$

(b) *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\chi \in \mathcal{S}(\mathbb{R}^n)$). Then*

$$\mathbb{R}^n \times \mathbb{R}^n \ni (y, \xi) \rightarrow \widehat{u\tau_y\chi}(\xi) = \left\langle u, e^{-i\langle \cdot, \xi \rangle} \chi(\cdot - y) \right\rangle \in \mathbb{C}$$

is a \mathcal{C}^∞ -function.

Proof. Let $q : \mathbb{R}_x^n \times \mathbb{R}_\xi^n \rightarrow \mathbb{R}$, $q(x, \xi) = \langle x, \xi \rangle$. Then $e^{-iq}(u \otimes 1) \in \mathcal{S}'(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$.

If $\varphi \in \mathcal{S}(\mathbb{R}_\xi^n)$, then we have

$$\begin{aligned} \langle e^{-iq}(u \otimes 1), \chi \otimes \varphi \rangle &= \langle u \otimes 1, e^{-iq}(\chi \otimes \varphi) \rangle \\ &= \left\langle u(x), \left\langle 1(\xi), e^{-iq(x, \xi)} \chi(x) \varphi(\xi) \right\rangle \right\rangle \\ &= \left\langle u(x), \chi(x) \left\langle 1(\xi), e^{-i\langle x, \xi \rangle} \varphi(\xi) \right\rangle \right\rangle \\ &= \langle u, \chi \widehat{\varphi} \rangle = \langle \widehat{\chi}u, \varphi \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \widehat{\chi}u, \varphi \rangle &= \langle e^{-iq}(u \otimes 1), \chi \otimes \varphi \rangle \\ &= \left\langle 1(\xi), \left\langle u(x), e^{-i\langle x, \xi \rangle} \chi(x) \varphi(\xi) \right\rangle \right\rangle \\ &= \left\langle 1(\xi), \varphi(\xi) \left\langle u, e^{-i\langle \cdot, \xi \rangle} \chi \right\rangle \right\rangle \\ &= \left\langle 1(\xi), \varphi(\xi) \left\langle e^{-i\langle \cdot, \xi \rangle} u, \chi \right\rangle \right\rangle \\ &= \int \varphi(\xi) \left\langle e^{-i\langle \cdot, \xi \rangle} u, \chi \right\rangle d\xi \end{aligned}$$

This proves that

$$\widehat{\chi}u(\xi) = \left\langle e^{-i\langle \cdot, \xi \rangle} u, \chi \right\rangle, \quad \xi \in \mathbb{R}^n.$$

□

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ (or $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus 0$). Let $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$) and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. By using (3.1) we get

$$\begin{aligned} \langle u\tau_z\tilde{\chi}, \varphi \rangle &= \frac{1}{\|\chi\|_{L^2}^2} \int \langle u\tau_z\tilde{\chi}, (\tau_y\chi)(\tau_y\bar{\chi})\varphi \rangle dy \\ &= \frac{1}{\|\chi\|_{L^2}^2} \int \langle u\tau_y\chi, (\tau_z\tilde{\chi})(\tau_y\bar{\chi})\varphi \rangle dy, \end{aligned}$$

$$|\langle u\tau_z\tilde{\chi}, \varphi \rangle| \leq \frac{1}{\|\chi\|_{L^2}^2} \int \|u\tau_y\chi\|_{\mathcal{B}_k} \|(\tau_z\tilde{\chi})(\tau_y\bar{\chi})\varphi\|_{\mathcal{B}_{1/k}} dy.$$

Let $\Gamma \subset \mathbb{R}^n$ be a lattice. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\chi \in \mathcal{S}(\mathbb{R}^n)$) be such that

$$\Psi = \Psi_{\Gamma, \chi} = \sum_{\gamma \in \Gamma} |\tau_\gamma \chi|^2 > 0.$$

Then $\Psi, \frac{1}{\Psi} \in \mathcal{BC}^\infty(\mathbb{R}^n)$ and both are Γ -periodic. Let $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$). Using (3.3) we obtain that

$$\begin{aligned} \langle u\tau_z\tilde{\chi}, \varphi \rangle &= \sum_{\gamma \in \Gamma} \left\langle u\tau_\gamma\chi, \frac{1}{\Psi}(\tau_\gamma\bar{\chi})(\tau_z\tilde{\chi})\varphi \right\rangle, \\ |\langle u\tau_z\tilde{\chi}, \varphi \rangle| &\leq \sum_{\gamma \in \Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k} \left\| \frac{1}{\Psi}(\tau_\gamma\bar{\chi})(\tau_z\tilde{\chi})\varphi \right\|_{\mathcal{B}_{1/k}} \\ &\leq C_\Psi \sum_{\gamma \in \Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k} \|(\tau_\gamma\bar{\chi})(\tau_z\tilde{\chi})\varphi\|_{\mathcal{B}_{1/k}}. \end{aligned}$$

In the last inequality we used the Proposition 2.7 and the fact that $\frac{1}{\Psi} \in \mathcal{BC}^\infty(\mathbb{R}^n)$.

If (Y, μ) is either \mathbb{R}^n with Lebesgue measure or Γ with the counting measure, then the previous estimates can be written as:

$$|\langle u\tau_z\tilde{\chi}, \varphi \rangle| \leq Cst \int_Y \|u\tau_y\chi\|_{\mathcal{B}_k} \|(\tau_z\tilde{\chi})(\tau_y\bar{\chi})\varphi\|_{\mathcal{B}_{1/k}} d\mu(y)$$

We shall use Proposition 2.7 to estimate $\|(\tau_z\tilde{\chi})(\tau_y\bar{\chi})\varphi\|_{\mathcal{B}_{1/k}}$. Let us write m_k for $[N + \frac{n+1}{2}] + 1$. Then we have

$$\|(\tau_z\tilde{\chi})(\tau_y\bar{\chi})\varphi\|_{\mathcal{B}_{1/k}} \leq Cst \sup_{|\alpha+\beta| \leq m_k} |((\tau_z\partial^\alpha\tilde{\chi})(\tau_y\partial^\beta\bar{\chi}))| \|\varphi\|_{\mathcal{B}_{1/k}}.$$

There is a continuous seminorm $p = p_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ so that

$$\begin{aligned} |(\tau_z\partial^\alpha\tilde{\chi})(\tau_y\partial^\beta\bar{\chi})(x)| &\leq p(\tilde{\chi})p(\chi)\langle x-z \rangle^{-2(n+1)}\langle x-y \rangle^{-2(n+1)} \\ &\leq 2^{n+1}p(\tilde{\chi})p(\chi)\langle 2x-z-y \rangle^{-n-1}\langle z-y \rangle^{-n-1} \\ &\leq 2^{n+1}p(\tilde{\chi})p(\chi)\langle z-y \rangle^{-n-1}, \quad |\alpha+\beta| \leq m_k. \end{aligned}$$

Here we used the inequality

$$\langle X \rangle^{-2(n+1)} \langle Y \rangle^{-2(n+1)} \leq 2^{n+1} \langle X+Y \rangle^{-n-1} \langle X-Y \rangle^{-n-1}, \quad X, Y \in \mathbb{R}^m$$

which is a consequence of Peetre's inequality:

$$\begin{aligned} \langle X + Y \rangle^{n+1} &\leq 2^{\frac{n+1}{2}} \langle X \rangle^{n+1} \langle Y \rangle^{n+1} \\ \langle X - Y \rangle^{n+1} &\leq 2^{\frac{n+1}{2}} \langle X \rangle^{n+1} \langle Y \rangle^{n+1} \\ &\Downarrow \\ \langle X + Y \rangle^{n+1} \langle X - Y \rangle^{n+1} &\leq 2^{n+1} \langle X \rangle^{2(n+1)} \langle Y \rangle^{2(n+1)} \end{aligned}$$

Hence

$$\begin{aligned} \sup_{|\alpha+\beta|\leq m_k} |((\tau_z \partial^\alpha \tilde{\chi}) (\tau_y \partial^\beta \bar{\chi}))| &\leq 2^{n+1} p_{n,k}(\tilde{\chi}) p_{n,k}(\chi) \langle z - y \rangle^{-n-1}, \\ \|(\tau_z \tilde{\chi}) (\tau_y \bar{\chi}) \varphi\|_{\mathcal{B}_{1/k}} &\leq C(n, k, \chi, \tilde{\chi}) \langle z - y \rangle^{-n-1} \|\varphi\|_{\mathcal{B}_{1/k}}, \\ |\langle u \tau_z \tilde{\chi}, \varphi \rangle| &\leq C(n, k, \chi, \tilde{\chi}) \left(\int_Y \|u \tau_y \chi\|_{\mathcal{B}_k} \langle z - y \rangle^{-n-1} d\mu(y) \right) \|\varphi\|_{\mathcal{B}_{1/k}}. \end{aligned}$$

The last estimate implies that

$$\|u \tau_z \tilde{\chi}\|_{\mathcal{B}_k} \leq C(n, k, \chi, \tilde{\chi}) \left(\int_Y \|u \tau_y \chi\|_{\mathcal{B}_k} \langle z - y \rangle^{-n-1} d\mu(y) \right)$$

Let $1 \leq p < \infty$. If (Z, ν) is either \mathbb{R}^n with Lebesgue measure or a lattice with the counting measure, then Schur's lemma implies

$$\left(\int_Z \|u \tau_z \tilde{\chi}\|_{\mathcal{B}_k}^p d\nu(z) \right)^{\frac{1}{p}} \leq C'(n, k, \chi, \tilde{\chi}) \left\| \langle \cdot \rangle^{-n-1} \right\|_{L^1} \left(\int_Y \|u \tau_y \chi\|_{\mathcal{B}_k}^p d\mu(y) \right)^{\frac{1}{p}}$$

For $p = \infty$ we have

$$\sup_z \|u \tau_z \tilde{\chi}\|_{\mathcal{B}_k} \leq C'(n, k, \chi, \tilde{\chi}) \left\| \langle \cdot \rangle^{-n-1} \right\|_{L^1} \sup_y \|u \tau_y \chi\|_{\mathcal{B}_k}.$$

By taking different combinations of (Y, μ) and (Z, ν) we obtain the following result.

Proposition 3.4. *Let $k \in \mathcal{K}(\mathbb{R}^n)$ and C, N the positive constants that define k and $1 \leq p < \infty$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ (or $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus 0$).*

(a) *If $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$), then there is $C(n, k, \chi, \tilde{\chi}) > 0$ such that*

$$\begin{aligned} \left(\int \|u \tau_{\tilde{y}} \tilde{\chi}\|_{\mathcal{B}_k}^p d\tilde{y} \right)^{\frac{1}{p}} &\leq C(n, k, \chi, \tilde{\chi}) \left(\int \|u \tau_y \chi\|_{\mathcal{B}_k}^p dy \right)^{\frac{1}{p}}, \\ \sup_{\tilde{y}} \|u \tau_{\tilde{y}} \tilde{\chi}\|_{\mathcal{B}_k} &\leq C(n, k, \chi, \tilde{\chi}) \sup_y \|u \tau_y \chi\|_{\mathcal{B}_k}. \end{aligned}$$

(b) *If $\Gamma \subset \mathbb{R}^n$ is a lattice such that*

$$\Psi = \Psi_{\Gamma, \chi} = \sum_{\gamma \in \Gamma} |\tau_\gamma \chi|^2 > 0$$

and $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$), then there is $C(n, k, \Gamma, \chi, \tilde{\chi}) > 0$ such that

$$\begin{aligned} \left(\int \|u \tau_{\tilde{y}} \tilde{\chi}\|_{\mathcal{B}_k}^p d\tilde{y} \right)^{\frac{1}{p}} &\leq C(n, k, \Gamma, \chi, \tilde{\chi}) \left(\sum_{\gamma \in \Gamma} \|u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}}, \\ \sup_{\tilde{y}} \|u \tau_{\tilde{y}} \tilde{\chi}\|_{\mathcal{B}_k} &\leq C(n, k, \Gamma, \chi, \tilde{\chi}) \sup_{\gamma} \|u \tau_\gamma \chi\|_{\mathcal{B}_k}. \end{aligned}$$

(c) If $\tilde{\Gamma} \subset \mathbb{R}^n$ is a lattice and $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$), then there is $C(n, k, \tilde{\Gamma}, \chi, \tilde{\chi}) > 0$ such that

$$\left(\sum_{\tilde{\gamma} \in \tilde{\Gamma}} \|u\tau_{\tilde{\gamma}}\tilde{\chi}\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \leq C(n, k, \tilde{\Gamma}, \chi, \tilde{\chi}) \left(\int \|u\tau_y\chi\|_{\mathcal{B}_k}^p dy \right)^{\frac{1}{p}},$$

$$\sup_{\tilde{\gamma}} \|u\tau_{\tilde{\gamma}}\tilde{\chi}\|_{\mathcal{B}_k} \leq C(n, k, \tilde{\Gamma}, \chi, \tilde{\chi}) \sup_y \|u\tau_y\chi\|_{\mathcal{B}_k}.$$

(d) If $\Gamma, \tilde{\Gamma} \subset \mathbb{R}^n$ are lattices such that

$$\Psi = \Psi_{\Gamma, \chi} = \sum_{\gamma \in \Gamma} |\tau_\gamma \chi|^2 > 0$$

and $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$), then there is $C(n, k, \Gamma, \tilde{\Gamma}, \chi, \tilde{\chi}) > 0$ such that

$$\left(\sum_{\tilde{\gamma} \in \tilde{\Gamma}} \|u\tau_{\tilde{\gamma}}\chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \leq C(n, k, \Gamma, \tilde{\Gamma}, \chi, \tilde{\chi}) \left(\sum_{\gamma \in \Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}},$$

$$\sup_{\tilde{\gamma}} \|u\tau_{\tilde{\gamma}}\chi\|_{\mathcal{B}_k} \leq C(n, k, \Gamma, \tilde{\Gamma}, \chi, \tilde{\chi}) \sup_{\gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}.$$

Definition 3.5. Let $1 \leq p \leq \infty$, $k \in \mathcal{K}(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$. We say that u belongs to $\mathcal{B}_k^p(\mathbb{R}^n)$ if there is $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ such that the measurable function $\mathbb{R}^n \ni y \rightarrow \|u\tau_y\chi\|_{\mathcal{B}_k} \in \mathbb{R}$ belongs to L^p . We put

$$\|u\|_{k, p, \chi} = \left(\int \|u\tau_y\chi\|_{\mathcal{B}_k}^p dy \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|u\|_{k, \infty, \chi} \equiv \|u\|_{k, \text{ul}, \chi} = \sup_y \|u\tau_y\chi\|_{\mathcal{B}_k}.$$

Proposition 3.6. (a) The above definition does not depend on the choice of the function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$.

(b) If $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$, then $\|\cdot\|_{k, p, \chi}$ is a norm on $\mathcal{B}_k^p(\mathbb{R}^n)$ and the topology that defines does not depend on the function χ .

(c) Let $\Gamma \subset \mathbb{R}^n$ be a lattice and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a function with the property that

$$\Psi = \Psi_{\Gamma, \chi} = \sum_{\gamma \in \Gamma} |\tau_\gamma \chi|^2 > 0.$$

Then

$$\mathcal{B}_k^p(\mathbb{R}^n) \ni u \rightarrow \begin{cases} \left(\sum_{\gamma \in \Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup_{\gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k} & p = \infty \end{cases}$$

is a norm on $\mathcal{B}_k^p(\mathbb{R}^n)$ and the topology that defines is the topology of $\mathcal{B}_k^p(\mathbb{R}^n)$. We shall use the notation

$$\|u\|_{k, p, \Gamma, \chi} = \begin{cases} \left(\sum_{\gamma \in \Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup_{\gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k} & p = \infty \end{cases}.$$

(d) If $1 \leq p \leq q \leq \infty$, Then

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{B}_k^1(\mathbb{R}^n) \subset \mathcal{B}_k^p(\mathbb{R}^n) \subset \mathcal{B}_k^q(\mathbb{R}^n) \subset \mathcal{B}_k^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

- (e) If $k', k \in \mathcal{K}(\mathbb{R}^n)$ and $k' \leq Cst \cdot k$, then $\mathcal{B}_k^p(\mathbb{R}^n) \subset \mathcal{B}_{k'}^p(\mathbb{R}^n)$.
- (f) $(\mathcal{B}_k^p(\mathbb{R}^n), \|\cdot\|_{k,p,\chi})$ is a Banach space.
- (g) If $1/k \in L^2(\mathbb{R}^n)$, then $\mathcal{B}_k^\infty(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$.

Proof. (a) (b) (c) are immediate consequences of the previous proposition.

(d) The inclusions $\mathcal{B}_k^1(\mathbb{R}^n) \subset \mathcal{B}_k^p(\mathbb{R}^n) \subset \mathcal{B}_k^q(\mathbb{R}^n) \subset \mathcal{B}_k^\infty(\mathbb{R}^n)$ are consequences of the elementary inclusions $l^1 \subset l^p \subset l^q \subset l^\infty$. What remain to be shown are the inclusions $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{B}_k^1(\mathbb{R}^n)$, $\mathcal{B}_k^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Let $u \in \mathcal{B}_k^\infty(\mathbb{R}^n)$, $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We have

$$\begin{aligned}
 \langle u, \varphi \rangle &= \frac{1}{\|\chi\|_{L^2}^2} \int \langle u, (\tau_y \chi) (\tau_y \bar{\chi}) \varphi \rangle dy \\
 &= \frac{1}{\|\chi\|_{L^2}^2} \int \langle u \tau_y \chi, (\tau_y \bar{\chi}) \varphi \rangle dy, \\
 |\langle u, \varphi \rangle| &\leq \frac{1}{\|\chi\|_{L^2}^2} \int |\langle u \tau_y \chi, (\tau_y \bar{\chi}) \varphi \rangle| dy \\
 &\leq \frac{1}{\|\chi\|_{L^2}^2} \int \|u \tau_y \chi\|_{\mathcal{B}_k} \|(\tau_y \bar{\chi}) \varphi\|_{\mathcal{B}_{1/k}} dy \\
 &\leq \frac{1}{\|\chi\|_{L^2}^2} \|u\|_{k,\infty,\chi} \int \|(\tau_y \bar{\chi}) \varphi\|_{\mathcal{B}_{1/k}} dy
 \end{aligned}$$

We shall use Proposition 2.7 to estimate $\|(\tau_y \bar{\chi}) \varphi\|_{\mathcal{B}_{1/k}}$. Let $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\tilde{\chi} = 1$ on $\text{supp}\chi$. If $m_k = [N + \frac{n+1}{2}] + 1$, then we obtain that

$$\begin{aligned}
 \|(\tau_y \bar{\chi}) \varphi\|_{\mathcal{B}_{1/k}} &\leq C \sup_{|\alpha+\beta| \leq m_k} |(\partial^\alpha \varphi) (\tau_y \partial^\beta \bar{\chi})| \|\tau_y \tilde{\chi}\|_{\mathcal{B}_{1/k}} \\
 &= C \sup_{|\alpha+\beta| \leq m_k} |(\partial^\alpha \varphi) (\tau_y \partial^\beta \bar{\chi})| \|\tilde{\chi}\|_{\mathcal{B}_{1/k}}.
 \end{aligned}$$

Since $\chi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that there is a continuous seminorm $p = p_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ so that

$$\begin{aligned}
 |(\partial^\alpha \varphi) (\tau_y \partial^\beta \bar{\chi}) (x)| &\leq p(\varphi) p(\chi) \langle x - y \rangle^{-2(n+1)} \langle x \rangle^{-2(n+1)} \\
 &\leq 2^{n+1} p(\varphi) p(\chi) \langle 2x - y \rangle^{-(n+1)} \langle y \rangle^{-(n+1)} \\
 &\leq 2^{n+1} p(\varphi) p(\chi) \langle y \rangle^{-(n+1)}, \quad |\alpha + \beta| \leq m_k.
 \end{aligned}$$

Hence

$$|\langle u, \varphi \rangle| \leq 2^{n+1} C \frac{1}{\|\chi\|_{L^2}^2} \|u\|_{k,\infty,\chi} \left\| \langle \cdot \rangle^{-(n+1)} \right\|_{L^1} \|\tilde{\chi}\|_{\mathcal{B}_{1/k}} p(\chi) p(\varphi).$$

If $u \in \mathcal{S}(\mathbb{R}^n)$, $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ and $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\tilde{\chi} = 1$ on $\text{supp}\chi$, then Proposition 2.7 and the above arguments imply that there is a continuous seminorm $p = p_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ so that

$$\begin{aligned}
 \|(\tau_y \chi) u\|_{\mathcal{B}_k} &\leq C \sup_{|\alpha+\beta| \leq m_k} |(\partial^\alpha u) (\tau_y \partial^\beta \chi)| \|\tau_y \tilde{\chi}\|_{\mathcal{B}_k} \\
 &= C \sup_{|\alpha+\beta| \leq m_k} |(\partial^\alpha u) (\tau_y \partial^\beta \chi)| \|\tilde{\chi}\|_{\mathcal{B}_k} \\
 &\leq 2^{n+1} C p(u) p(\chi) \langle y \rangle^{-(n+1)} \|\tilde{\chi}\|_{\mathcal{B}_k}
 \end{aligned}$$

Hence $u \in \mathcal{B}_k^1(\mathbb{R}^n)$ and

$$\|u\|_{k,1,\chi} \leq Cst \left\| \langle \cdot \rangle^{-(n+1)} \right\|_{L^1} \|\tilde{\chi}\|_{\mathcal{B}_k} p(u) p(\chi)$$

(e) is trivial.

(f) Let $\{u_n\}$ be a Cauchy sequence in $\mathcal{B}_k^p(\mathbb{R}^n)$. Since $\mathcal{B}_k^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ is sequentially complete, there is $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $u_n \rightarrow u$ in $\mathcal{S}'(\mathbb{R}^n)$.

Let $\Gamma \subset \mathbb{R}^n$ be a lattice and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a function with the property that

$$\Psi = \Psi_{\Gamma,\chi} = \sum_{\gamma \in \Gamma} |\tau_\gamma \chi|^2 > 0.$$

Then for any $\gamma \in \Gamma$ there is $u_\gamma \in \mathcal{B}_k(\mathbb{R}^n)$ such that $u_n \tau_\gamma \chi \rightarrow u_\gamma$ in $\mathcal{B}_k(\mathbb{R}^n)$. As $u_n \rightarrow u$ in $\mathcal{S}'(\mathbb{R}^n)$, it follows that $u_\gamma = u \tau_\gamma \chi$ for any $\gamma \in \Gamma$.

Since $\{u_n\}$ is a Cauchy sequence in $\mathcal{B}_k^p(\mathbb{R}^n)$ there is $M \in (0, \infty)$ such that $\|u_n\|_{k,p,\Gamma,\chi} \leq M$ for any $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then there is n_ε such that if $m, n \geq n_\varepsilon$, then $\|u_m - u_n\|_{k,p,\Gamma,\chi} < \varepsilon$.

Let $F \subset \Gamma$ a finite subset. Then

$$\begin{aligned} \left(\sum_{\gamma \in F} \|u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} &\leq \left(\sum_{\gamma \in F} \|u \tau_\gamma \chi - u_n \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} + \left(\sum_{\gamma \in F} \|u_n \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{\gamma \in F} \|u \tau_\gamma \chi - u_n \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} + M \end{aligned}$$

By passing to the limit we obtain $\left(\sum_{\gamma \in F} \|u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \leq M$ for any $F \subset \Gamma$ a finite subset. Hence $u \in \mathcal{B}_k^p(\mathbb{R}^n)$.

For $F \subset \Gamma$ a finite subset and $m, n \geq n_\varepsilon$ we have

$$\begin{aligned} \left(\sum_{\gamma \in F} \|u \tau_\gamma \chi - u_n \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} &\leq \left(\sum_{\gamma \in F} \|u \tau_\gamma \chi - u_m \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} + \left(\sum_{\gamma \in F} \|u_n \tau_\gamma \chi - u_m \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{\gamma \in F} \|u \tau_\gamma \chi - u_m \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} + \varepsilon \end{aligned}$$

By letting $m \rightarrow \infty$ we obtain $\left(\sum_{\gamma \in F} \|u \tau_\gamma \chi - u_n \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \leq \varepsilon$ for any $F \subset \Gamma$ a finite subset and $n \geq n_\varepsilon$. This implies that $u_n \rightarrow u$ in $\mathcal{B}_k^p(\mathbb{R}^n)$. The case $p = \infty$ is even simpler.

(g) If $1/k \in L^2$, then $\mathcal{B}_k^\infty(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\chi(0) = 1$. Then for $x \in \mathbb{R}^n$

$$\begin{aligned} |u(x)| &= |u\tau_x\chi(x)| \leq (2\pi)^{-n} \|\widehat{u\tau_x\chi}\|_{L^1} \\ &\leq (2\pi)^{-n} \|1/k\|_{L^2} \|u\tau_x\chi\|_{\mathcal{B}_k} \\ &\leq (2\pi)^{-n} \|1/k\|_{L^2} \sup_y \|u\tau_y\chi\|_{\mathcal{B}_k} \\ &= (2\pi)^{-n} \|1/k\|_{L^2} \|u\|_{k,\infty,\chi}. \end{aligned}$$

□

Remark 3.7. The spaces $\mathcal{B}_k^p(\mathbb{R}^n)$ are particular cases of Wiener amalgam spaces. More precisely we have

$$\mathcal{B}_k^p(\mathbb{R}^n) = W(\mathcal{B}_k, L^p)$$

with local component $\mathcal{B}_k(\mathbb{R}^n)$ and global component L^p . Wiener amalgam spaces were introduced by Hans Georg Feichtinger in 1980.

Now using the techniques of Coifman and Meyer, developed for the study of Beurling algebras A_ω and B_ω (see [Co-Me] pp 7-10), we shall prove an interesting result.

Theorem 3.8 (localization principle). $\mathcal{B}_k(\mathbb{R}^n) = \mathcal{B}_k^2(\mathbb{R}^n) = W(\mathcal{B}_k, L^2)$.

To prove the result, we shall use the partition of unity built in the previous section. Let $m \in \mathbb{N}$ and $\{x_1, \dots, x_m\} \subset \mathbb{R}^n$ be such that

$$[0, 1]^n \subset \left(x_1 + \left[\frac{1}{3}, \frac{2}{3}\right]^n\right) \cup \dots \cup \left(x_m + \left[\frac{1}{3}, \frac{2}{3}\right]^n\right)$$

Let $\tilde{h} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\tilde{h} \geq 0$, be such that $\tilde{h} = 1$ on $[\frac{1}{3}, \frac{2}{3}]^n$ and $\text{supp}\tilde{h} \subset [\frac{1}{4}, \frac{3}{4}]^n$. Then

- (a) $\tilde{H} = \sum_{i=1}^m \sum_{\gamma \in \mathbb{Z}^n} \tau_{\gamma+x_i} \tilde{h} \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is \mathbb{Z}^n -periodic and $\tilde{H} \geq 1$.
- (b) $h_i = \frac{\tau_{x_i} \tilde{h}}{\tilde{H}} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h_i \geq 0$, $\text{supp}h_i \subset x_i + [\frac{1}{4}, \frac{3}{4}]^n = K_i$, $(K_i - K_i) \cap \mathbb{Z}^n = \{0\}$, $i = 1, \dots, m$.
- (c) $\chi_i = \sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma h_i \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is \mathbb{Z}^n -periodic, $i = 1, \dots, m$ and $\sum_{i=1}^m \chi_i = 1$.
- (d) $h = \sum_{i=1}^m h_i \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h \geq 0$, $\sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma h = 1$.

Lemma 3.9. $\mathcal{B}_k^2(\mathbb{R}^n) \subset \mathcal{B}_k(\mathbb{R}^n)$.

Proof. Let $u \in \mathcal{B}_k^2(\mathbb{R}^n)$. We have

$$u = \sum_{j=1}^m \chi_j u \quad \text{with} \quad \chi_j u = \sum_{\gamma \in \mathbb{Z}^n} (\tau_\gamma h_j) u.$$

Since $u \in \mathcal{B}_k^2(\mathbb{R}^n)$ applying Proposition 3.4 we get that

$$\sum_{\gamma \in \mathbb{Z}^n} \|(\tau_\gamma h_j) u\|_{\mathcal{B}_k}^2 < \infty.$$

Using Lemma 2.6 it follows that $\chi_j u \in \mathcal{B}_k(\mathbb{R}^n)$ and

$$\|\chi_j u\|_{\mathcal{B}_k} \approx \left(\sum_{\gamma \in \mathbb{Z}^n} \|(\tau_\gamma h_j) u\|_{\mathcal{B}_k}^2 \right)^{\frac{1}{2}} \leq C_j \|u\|_{k,2}$$

where $\|\cdot\|_{k,2}$ is a fixed norm on $\mathcal{B}_k^2(\mathbb{R}^n)$. So $u = \sum_{j=1}^m \chi_j u \in \mathcal{B}_k(\mathbb{R}^n)$ and

$$\|u\|_{\mathcal{B}_k} \leq \sum_{j=1}^m \|\chi_j u\|_{\mathcal{B}_k} \leq \left(\sum_{j=1}^m C_j \right) \|u\|_{k,2}.$$

□

Lemma 3.10. $\mathcal{B}_k(\mathbb{R}^n) \subset \mathcal{B}_k^2(\mathbb{R}^n)$.

Proof. Then the following statements are equivalent:

- (i) $u \in \mathcal{B}_k(\mathbb{R}^n)$
 - (ii) $\chi_j u \in \mathcal{B}_k(\mathbb{R}^n)$, $j = 1, \dots, m$. (Here we use Lemma 2.4 (b))
 - (iii) $\left\{ \|(\tau_\gamma h_j) u\|_{\mathcal{B}_k} \right\}_{\gamma \in \mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$, $j = 1, \dots, m$. (Here we use Lemma 2.6)
- Since $h = \sum_{j=1}^m h_j$ and

$$\|(\tau_\gamma h) u\|_{\mathcal{B}_k} \leq \sum_{j=1}^m \|(\tau_\gamma h_j) u\|_{\mathcal{B}_k}, \quad \gamma \in \mathbb{Z}^n$$

we get that $\left\{ \|(\tau_\gamma h) u\|_{\mathcal{B}_k} \right\}_{\gamma \in \mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$. Since $h = \sum_{j=1}^m h_j \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h \geq 0$, $\sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma h = 1$ it follows that $u \in \mathcal{B}_k^2(\mathbb{R}^n)$ and

$$\begin{aligned} \|u\|_{k,2,h} &\approx \left\| \left\{ \|(\tau_\gamma h) u\|_{\mathcal{B}_k} \right\}_{\gamma \in \mathbb{Z}^n} \right\|_{l^2(\mathbb{Z}^n)} \\ &\leq \sum_{j=1}^m \left\| \left\{ \|(\tau_\gamma h_j) u\|_{\mathcal{B}_k} \right\}_{\gamma \in \mathbb{Z}^n} \right\|_{l^2(\mathbb{Z}^n)} \\ &\approx \sum_{j=1}^m \|\chi_j u\|_{\mathcal{B}_k} \leq Cst \|u\|_{\mathcal{B}_k}. \end{aligned}$$

□

Lemma 3.11. If $1 \leq p < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{B}_k^p(\mathbb{R}^n)$.

Proof. (i) Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ on $B(0,1)$, $\psi^\varepsilon(x) = \psi(\varepsilon x)$, $0 < \varepsilon \leq 1$, $x \in \mathbb{R}^n$. If $u \in \mathcal{B}_k(\mathbb{R}^n)$, then $\psi^\varepsilon u \rightarrow u$ in $\mathcal{B}_k(\mathbb{R}^n)$. Moreover we have

$$\|\psi^\varepsilon u\|_{\mathcal{B}_k} \leq C(k, \psi) \|u\|_{\mathcal{B}_k}, \quad 0 < \varepsilon \leq 1,$$

where

$$\begin{aligned} C(k, \psi) &= (2\pi)^{-n} \sup_{0 < \varepsilon \leq 1} \left(\int M_k(\eta) \varepsilon^{-n} |\widehat{\psi}(\eta/\varepsilon)| d\eta \right) \\ &= (2\pi)^{-n} 2^{N/2} (\max\{1, C\})^N \sup_{0 < \varepsilon \leq 1} \left(\int \langle \varepsilon \eta \rangle^N |\widehat{\psi}(\eta)| d\eta \right) \\ &\leq (2\pi)^{-n} 2^{N/2} (\max\{1, C\})^N \left(\int \langle \eta \rangle^N |\widehat{\psi}(\eta)| d\eta \right). \end{aligned}$$

(ii) Suppose that $u \in \mathcal{B}_k^p(\mathbb{R}^n)$. Let $F \subset \mathbb{Z}^n$ be an arbitrary finite subset. Then the subadditivity property of the norm $\|\cdot\|_{l^p}$ implies that:

$$\begin{aligned} \|\psi^\varepsilon u - u\|_{k,p,\mathbb{Z}^n,\chi} &\leq \left(\sum_{\gamma \in F} \|\psi^\varepsilon u \tau_\gamma \chi - u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} + \left(\sum_{\gamma \in \mathbb{Z}^n \setminus F} \|\psi^\varepsilon u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \\ &\quad + \left(\sum_{\gamma \in \mathbb{Z}^n \setminus F} \|u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{\gamma \in F} \|\psi^\varepsilon u \tau_\gamma \chi - u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} + (C(k, \psi) + 1) \left(\sum_{\gamma \in \mathbb{Z}^n \setminus F} \|u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \end{aligned}$$

By making $\varepsilon \rightarrow 0$ we deduce that

$$\limsup_{\varepsilon \rightarrow 0} \|\psi^\varepsilon u - u\|_{k,p,\mathbb{Z}^n,\chi} \leq (C(k, \psi) + 1) \left(\sum_{\gamma \in \mathbb{Z}^n \setminus F} \|u \tau_\gamma \chi\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}}$$

for any $F \subset \mathbb{Z}^n$ finite subset. Hence $\lim_{\varepsilon \rightarrow 0} \psi^\varepsilon u = u$ in $\mathcal{B}_k^p(\mathbb{R}^n)$. The immediate consequence is that

(iii) $\mathcal{E}'(\mathbb{R}^n) \cap \mathcal{B}_k^p(\mathbb{R}^n)$ is dense in $\mathcal{B}_k^p(\mathbb{R}^n)$.

(iv) Suppose that $u \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{B}_k^p(\mathbb{R}^n)$. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\text{supp } \varphi \subset B(0; 1)$, $\int \varphi(x) dx = 1$. For $\varepsilon \in (0, 1]$, we set $\varphi_\varepsilon = \varepsilon^{-n} \varphi(\cdot/\varepsilon)$. Let $K = \text{supp } u + \overline{B(0; 1)}$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that

$$\Psi = \Psi_{\Gamma, \chi} = \sum_{\gamma \in \Gamma} |\tau_\gamma \chi|^2 > 0.$$

Then there is a finite set $F = F_{K, \chi} \subset \mathbb{Z}^n$ such that $(\tau_\gamma \chi)(\varphi_\varepsilon * u - u) = 0$ for any $\gamma \in \mathbb{Z}^n \setminus F$. It follows that

$$\begin{aligned} \|\varphi_\varepsilon * u - u\|_{k,p,\mathbb{Z}^n,\chi} &= \left(\sum_{\gamma \in F} \|(\tau_\gamma \chi)(\varphi_\varepsilon * u - u)\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \\ &\approx \left(\sum_{\gamma \in F} \|(\tau_\gamma \chi)(\varphi_\varepsilon * u - u)\|_{\mathcal{B}_k}^2 \right)^{\frac{1}{2}} \\ &\approx \|\varphi_\varepsilon * u - u\|_{\mathcal{B}_k} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

□

Proposition 3.12. Let $1 \leq p \leq \infty$, $k \in \mathcal{K}(\mathbb{R}^n)$ and C, N the positive constants that define k . Let $m_k = \lceil N + \frac{n+1}{2} \rceil + 1$. Then

$$\mathcal{BC}^{m_k}(\mathbb{R}^n) \cdot \mathcal{B}_k^p(\mathbb{R}^n) \subset \mathcal{B}_k^p(\mathbb{R}^n).$$

Proof. Let $u \in \mathcal{B}_k^p(\mathbb{R}^n)$ and $\psi \in \mathcal{BC}^{m_k}(\mathbb{R}^n)$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$. By using Proposition 2.7 we obtain that $\psi u \tau_y \chi^2 \in \mathcal{B}_k(\mathbb{R}^n)$ and

$$\begin{aligned} \|\psi u \tau_y \chi^2\|_{\mathcal{B}_k} &\leq C_k \|\psi \tau_y \chi\|_{\mathcal{BC}^{m_k}} \|u \tau_y \chi\|_{\mathcal{B}_k} \\ &\leq C_{k, \chi} \|\psi\|_{\mathcal{BC}^{m_k}} \|u \tau_y \chi\|_{\mathcal{B}_k} \end{aligned}$$

This inequality implies that

$$\|\psi u\|_{k,p,\chi^2} \leq C_{k,\chi} \|\psi\|_{\mathcal{BC}^{m_k}} \|u\|_{k,p,\chi}$$

□

Corollary 3.13. *Let $1 \leq p \leq \infty$ and $k \in \mathcal{K}(\mathbb{R}^n)$. Then*

$$\mathcal{BC}^\infty(\mathbb{R}^n) \cdot \mathcal{B}_k^p(\mathbb{R}^n) \subset \mathcal{B}_k^p(\mathbb{R}^n).$$

Corollary 3.14. *Let $k \in \mathcal{K}(\mathbb{R}^n)$ and C, N the positive constants that define k . Let $m_k = \lceil N + \frac{n+1}{2} \rceil + 1$. Then*

$$\mathcal{BC}^{m_k}(\mathbb{R}^n) \subset \mathcal{B}_k^\infty(\mathbb{R}^n).$$

Proof. Since $1 \in \mathcal{B}_k^\infty(\mathbb{R}^n)$ it follows that

$$\mathcal{BC}^{m_k}(\mathbb{R}^n) = \mathcal{BC}^{m_k}(\mathbb{R}^n) \cdot 1 \subset \mathcal{BC}^{m_k}(\mathbb{R}^n) \cdot \mathcal{B}_k^\infty(\mathbb{R}^n) \subset \mathcal{B}_k^\infty(\mathbb{R}^n).$$

□

4. WEAK WIENER-LÉVY THEOREM FOR \mathcal{B}_k^∞ ALGEBRAS

Lemma 4.1. *Let $k, k_1, k_2 \in \mathcal{K}(\mathbb{R}^n)$. Suppose that there is $C > 0$ such that*

$$\frac{1}{k_1^2} * \frac{1}{k_2^2} \leq \frac{C^2}{k^2},$$

i.e.

$$\int \frac{k^2(\xi)}{k_1^2(\eta) k_2^2(\xi - \eta)} d\eta \leq C^2.$$

Then the bilinear map

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (u_1, u_2) \rightarrow u_1 u_2 \in \mathcal{S}(\mathbb{R}^n)$$

has a bounded extension

$$\begin{aligned} \mathcal{B}_{k_1}(\mathbb{R}^n) \times \mathcal{B}_{k_2}(\mathbb{R}^n) &\ni (u_1, u_2) \rightarrow u_1 u_2 \in \mathcal{B}_k(\mathbb{R}^n), \\ \|u_1 u_2\|_{\mathcal{B}_k} &\leq (2\pi)^{-n} C \|u_1\|_{\mathcal{B}_{k_1}} \|u_2\|_{\mathcal{B}_{k_2}}. \end{aligned}$$

Proof. Let $(u_1, u_2) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned} \|u_1 u_2\|_{\mathcal{B}_k}^2 &= \|k \widehat{u_1 u_2}\|_{L^2}^2 = (2\pi)^{-2n} \|k(\widehat{u_1} * \widehat{u_2})\|_{L^2}^2 \\ &= (2\pi)^{-2n} \int |k(\xi) (\widehat{u_1} * \widehat{u_2})(\xi)|^2 d\xi \end{aligned}$$

By using Schwarz's inequality, we can estimate the integrand as follows

$$\begin{aligned} |k(\xi) (\widehat{u_1} * \widehat{u_2})(\xi)|^2 &\leq \left(\int |k_1(\eta) \widehat{u_1}(\eta)| |k_2(\xi - \eta) \widehat{u_2}(\xi - \eta)| \frac{k(\xi)}{k_1(\eta) k_2(\xi - \eta)} d\eta \right)^2 \\ &\leq \left(\int |k_1(\eta) \widehat{u_1}(\eta)|^2 |k_2(\xi - \eta) \widehat{u_2}(\xi - \eta)|^2 d\eta \right) \\ &\quad \cdot \left(\int \frac{k^2(\xi)}{k_1^2(\eta) k_2^2(\xi - \eta)} d\eta \right) \\ &\leq C^2 \left(\int |k_1(\eta) \widehat{u_1}(\eta)|^2 |k_2(\xi - \eta) \widehat{u_2}(\xi - \eta)|^2 d\eta \right) \end{aligned}$$

Hence

$$\begin{aligned} \|k(\widehat{u_1} * \widehat{u_2})\|_{L^2}^2 &\leq C^2 \int \left(\int |k_1(\eta) \widehat{u_1}(\eta)|^2 |k_2(\xi - \eta) \widehat{u_2}(\xi - \eta)|^2 d\eta \right) d\xi \\ &= C^2 \|u_1\|_{\mathcal{B}_{k_1}}^2 \|u_2\|_{\mathcal{B}_{k_2}}^2 \end{aligned}$$

and

$$\|u_1 u_2\|_{\mathcal{B}_k}^2 \leq (2\pi)^{-2n} C^2 \|u_1\|_{\mathcal{B}_{k_1}}^2 \|u_2\|_{\mathcal{B}_{k_2}}^2.$$

□

Corollary 4.2. *Let $k, k_1, k_2 \in \mathcal{K}(\mathbb{R}^n)$. Suppose that there is $C > 0$ such that*

$$\frac{1}{k_1^2} * \frac{1}{k_2^2} \leq \frac{C^2}{k^2},$$

i.e.

$$\int \frac{k^2(\xi)}{k_1^2(\eta) k_2^2(\xi - \eta)} d\eta \leq C^2.$$

If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, then

$$\mathcal{B}_{k_1}^{p_1}(\mathbb{R}^n) \cdot \mathcal{B}_{k_2}^{p_2}(\mathbb{R}^n) \subset \mathcal{B}_k^p(\mathbb{R}^n).$$

Proof. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$, $u_1 \in \mathcal{B}_{k_1}^{p_1}(\mathbb{R}^n)$ and $u_2 \in \mathcal{B}_{k_2}^{p_2}(\mathbb{R}^n)$. By using the previous lemma we obtain that $u_1 u_2 \tau_y \chi^2 \in \mathcal{B}_k(\mathbb{R}^n)$ and

$$\|u_1 u_2 \tau_y \chi^2\|_{\mathcal{B}_k} \leq (2\pi)^{-n} C \|u_1 \tau_y \chi\|_{\mathcal{B}_{k_1}} \|u_2 \tau_y \chi\|_{\mathcal{B}_{k_2}}$$

Finally, Hölder's inequality implies that

$$\|u_1 u_2\|_{k,p,\chi^2} \leq (2\pi)^{-n} C \|u_1\|_{k_1,p_1,\chi} \|u_2\|_{s,p_2,\chi}$$

□

Corollary 4.3. *Let $k \in \mathcal{K}(\mathbb{R}^n)$ and $1 \leq p \leq \infty$. Suppose that there is $C > 0$ such that*

$$\frac{1}{k^2} * \frac{1}{k^2} \leq \frac{C^2}{k^2}$$

i.e.

$$\int \frac{k^2(\xi)}{k^2(\eta) k^2(\xi - \eta)} d\eta \leq C^2.$$

Then $\mathcal{B}_k^\infty(\mathbb{R}^n)$ is a Banach algebra with respect to the usual product and $\mathcal{B}_k^p(\mathbb{R}^n)$ is an ideal in $\mathcal{B}_k^\infty(\mathbb{R}^n)$.

Definition 4.4. *The set of all temperate weights k satisfying $\frac{1}{k^2} * \frac{1}{k^2} \leq \frac{C_k^2}{k^2}$ will be denoted by $\mathcal{K}_a(\mathbb{R}^n)$. Then for any $k \in \mathcal{K}_a(\mathbb{R}^n)$, $\mathcal{B}_k^\infty(\mathbb{R}^n)$ is a Banach algebra with respect to the usual product.*

Lemma 4.5. *Let $k \in \mathcal{K}(\mathbb{R}^n)$. The map*

$$\mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{B}_k^\infty(\mathbb{R}^n) \ni (\varphi, u) \rightarrow \varphi * u \in \mathcal{B}_k^\infty(\mathbb{R}^n)$$

is well defined and for any $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus 0$ we have the estimate

$$\|\varphi * u\|_{k,\infty,\chi} \leq \|\varphi\|_{L^1} \|u\|_{k,\infty,\chi}, \quad (\varphi, u) \in \mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{B}_k^\infty(\mathbb{R}^n).$$

Proof. Let $(\varphi, u) \in \mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{B}_k^\infty(\mathbb{R}^n)$, $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus 0$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then using (3.2) we obtain

$$\begin{aligned} \langle \tau_z \chi (\varphi * u), \psi \rangle &= \langle u, \check{\varphi} * ((\tau_z \chi) \psi) \rangle = \int \check{\varphi}(y) \langle u, \tau_y ((\tau_z \chi) \psi) \rangle dy \\ &= \int \varphi(y) \langle u, \tau_{-y} ((\tau_z \chi) \psi) \rangle dy = \int \varphi(y) \langle (\tau_{z-y} \chi) u, \tau_{-y} \psi \rangle dy, \end{aligned}$$

where $\check{\varphi}(y) = \varphi(-y)$. Since

$$|\langle (\tau_{z-y} \chi) u, \tau_{-y} \psi \rangle| \leq \|(\tau_{z-y} \chi) u\|_{\mathcal{B}_k} \|\tau_{-y} \psi\|_{\mathcal{B}_{1/k}} \leq \|u\|_{k, \infty, \chi} \|\psi\|_{\mathcal{B}_{1/k}}$$

it follows that

$$|\langle \tau_z \chi (\varphi * u), \psi \rangle| \leq \|\varphi\|_{L^1} \|u\|_{k, \infty, \chi} \|\psi\|_{\mathcal{B}_{1/k}}$$

Hence $\tau_z \chi (\varphi * u) \in \mathcal{B}_k(\mathbb{R}^n)$ and $\|\tau_z \chi (\varphi * u)\|_{\mathcal{B}_k} \leq \|\varphi\|_{L^1} \|u\|_{k, \infty, \chi}$ for every $z \in \mathbb{R}^n$, i.e. $\varphi * u \in \mathcal{B}_k^\infty(\mathbb{R}^n)$ and

$$\|\varphi * u\|_{k, \infty, \chi} \leq \|\varphi\|_{L^1} \|u\|_{k, \infty, \chi}.$$

□

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\varphi \geq 0$ be such that $\text{supp} \varphi \subset B(0; 1)$, $\int \varphi(x) dx = 1$. For $\varepsilon \in (0, 1]$, we set $\varphi_\varepsilon = \varepsilon^{-n} \varphi(\cdot/\varepsilon)$.

Lemma 4.6. *If $k, k' \in \mathcal{K}(\mathbb{R}^n)$ and*

$$\frac{k'(\xi)}{k(\xi)} \rightarrow 0, \quad \xi \rightarrow \infty,$$

it follows that $\mathcal{B}_k^\infty(\mathbb{R}^n) \subset \mathcal{B}_{k'}^\infty(\mathbb{R}^n)$ and for any $u \in \mathcal{B}_{k'}^\infty(\mathbb{R}^n)$

$$\varphi_\varepsilon * u \rightarrow u \quad \text{in } \mathcal{B}_{k'}^\infty(\mathbb{R}^n), \quad \varepsilon \rightarrow 0.$$

Proof. Let $\chi, \chi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ be such that $\chi_0 = 1$ on $\text{supp} \chi + B(0; 1)$ and let $u \in \mathcal{B}_{k'}^\infty(\mathbb{R}^n)$. Then for $0 < \varepsilon \leq 1$ we have

$$(\varphi_\varepsilon * u - u) \tau_y \chi = (\varphi_\varepsilon * (u \tau_y \chi_0) - u \tau_y \chi_0) \tau_y \chi$$

and by Lemma 2.7

$$\|(\varphi_\varepsilon * u - u) \tau_y \chi\|_{\mathcal{B}_{k'}} \leq C(k', \chi) \|\varphi_\varepsilon * (u \tau_y \chi_0) - u \tau_y \chi_0\|_{\mathcal{B}_{k'}}.$$

We have

$$\begin{aligned} \|\varphi_\varepsilon * (u \tau_y \chi_0) - u \tau_y \chi_0\|_{\mathcal{B}_{k'}}^2 &= \|k' \mathcal{F}(\varphi_\varepsilon * (u \tau_y \chi_0) - u \tau_y \chi_0)\|_{L^2}^2 \\ &= \int |\widehat{\varphi}(\varepsilon \xi) - 1|^2 \left(\frac{k'(\xi)}{k(\xi)} \right)^2 |k(\xi) \widehat{u \tau_y \chi_0}(\xi)|^2 d\xi \end{aligned}$$

Given any $\delta > 0$ we now choose a ball $S = S_\delta$ so large that

$$|\widehat{\varphi}(\varepsilon \xi) - 1| \frac{k'(\xi)}{k(\xi)} \leq 2 \frac{k'(\xi)}{k(\xi)} < \delta, \quad \xi \in \mathbb{R}^n \setminus S, \quad \varepsilon \in (0, 1]$$

For $S = S_\delta$ we can choose ε_δ so small that $\varepsilon \in (0, \varepsilon_\delta]$ implies

$$\sup_{\xi \in S} |\widehat{\varphi}(\varepsilon \xi) - 1| \left(\frac{k'(\xi)}{k(\xi)} \right) < \delta.$$

By writing $\int = \int_S + \int_{\mathbb{R}^n \setminus S}$ we obtain

$$\begin{aligned} \|\varphi_\varepsilon * (u\tau_y\chi_0) - u\tau_y\chi_0\|_{\mathcal{B}_{k'}}^2 &\leq \delta^2 \int_S |k(\xi) \widehat{u\tau_y\chi_0}(\xi)|^2 d\xi \\ &\quad + \delta^2 \int_{\mathbb{R}^n \setminus S} |k(\xi) \widehat{u\tau_y\chi_0}(\xi)|^2 d\xi, \quad \varepsilon \in (0, \varepsilon_\delta], \end{aligned}$$

i.e.

$$\|\varphi_\varepsilon * (u\tau_y\chi_0) - u\tau_y\chi_0\|_{\mathcal{B}_{k'}} \leq \delta \|k \widehat{u\tau_y\chi_0}\|_{L^2} = \delta \|u\tau_y\chi_0\|_{\mathcal{B}_k}, \quad \varepsilon \in (0, \varepsilon_\delta].$$

It follows that

$$\|\varphi_\varepsilon * u - u\|_{k', \infty, \chi} \leq \delta C(k', \chi) \|u\|_{k, \infty, \chi_0}.$$

The proof is complete. \square

Definition 4.7. Let $k, k' \in \mathcal{K}(\mathbb{R}^n)$ be such that

$$k'(\xi) \leq Ck(\xi), \quad \xi \in \mathbb{R}^n.$$

We set $\mathcal{B}_{k(k')}^\infty(\mathbb{R}^n) \equiv (\mathcal{B}_k^\infty(\mathbb{R}^n), \|\cdot\|_{k', \infty})$.

Corollary 4.8. Let $k, k' \in \mathcal{K}(\mathbb{R}^n)$ be such that

$$\frac{k'(\xi)}{k(\xi)} \rightarrow 0, \quad \xi \rightarrow \infty.$$

Then

- (a) $\mathcal{B}_k^\infty(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$ is dense in $\mathcal{B}_{k(k')}^\infty(\mathbb{R}^n)$.
- (b) If $1/k \in L^2(\mathbb{R}^n)$, then $\mathcal{BC}^\infty(\mathbb{R}^n)$ is dense in $\mathcal{B}_{k(k')}^\infty(\mathbb{R}^n)$.

Proof. (b) If $1/k \in L^2$, then $\mathcal{B}_k^\infty(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$. Therefore, $\varphi_\varepsilon * \mathcal{B}_k^\infty(\mathbb{R}^n) \subset \varphi_\varepsilon * \mathcal{BC}(\mathbb{R}^n) \subset \mathcal{BC}^\infty(\mathbb{R}^n)$. \square

Theorem 4.9 (Wiener-Lévy for \mathcal{B}_k^∞ , weak form). Let $\Omega = \mathring{\Omega} \subset \mathbb{C}^d$ and $\Phi : \Omega \rightarrow \mathbb{C}$ a holomorphic function. Let $(k, k') \in \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}_a(\mathbb{R}^n)$ be such that

$$\frac{k'(\xi)}{k(\xi)} \rightarrow 0, \quad \xi \rightarrow \infty$$

and $1/k' \in L^2(\mathbb{R}^n)$.

- (a) If $u = (u_1, \dots, u_d) \in \mathcal{B}_k^\infty(\mathbb{R}^n)^d$ satisfies the condition $\overline{u(\mathbb{R}^n)} \subset \Omega$, then

$$\Phi \circ u \equiv \Phi(u) \in \mathcal{B}_{k'}^\infty(\mathbb{R}^n).$$

- (b) If $u, u_\varepsilon \in \mathcal{B}_k^\infty(\mathbb{R}^n)^d$, $0 < \varepsilon \leq 1$, $\overline{u(\mathbb{R}^n)} \subset \Omega$ and $u_\varepsilon \rightarrow u$ in $\mathcal{B}_{k'}^\infty(\mathbb{R}^n)^d$ as $\varepsilon \rightarrow 0$, then there is $\varepsilon_0 \in (0, 1]$ such that $\overline{u_\varepsilon(\mathbb{R}^n)} \subset \Omega$ for every $0 < \varepsilon \leq \varepsilon_0$ and $\Phi(u_\varepsilon) \rightarrow \Phi(u)$ in $\mathcal{B}_{k'}^\infty(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$.

Proof. On \mathbb{C}^d we shall consider the distance given by the norm

$$|z|_\infty = \max\{|z_1|, \dots, |z_d|\}, \quad z \in \mathbb{C}^d.$$

Let $r = \text{dist}(\overline{u(\mathbb{R}^n)}, \mathbb{C}^d \setminus \Omega)/8$. Since $\overline{u(\mathbb{R}^n)} \subset \Omega$ it follows that $r > 0$ and

$$\bigcup_{y \in \overline{u(\mathbb{R}^n)}} \overline{B(y; 4r)} \subset \Omega.$$

On $\mathcal{B}_{k'}^\infty(\mathbb{R}^n)^d$ we shall consider the norm

$$|||u|||_{k',\infty} = \max \left\{ \|u_1\|_{k',\infty}, \dots, \|u_d\|_{k',\infty} \right\}, \quad u \in \mathcal{B}_{k'}^\infty(\mathbb{R}^n)^d,$$

where $\|\cdot\|_{k',\infty}$ is a fixed Banach algebra norm on $\mathcal{B}_{k'}^\infty(\mathbb{R}^n)$, and on $\mathcal{BC}(\mathbb{R}^n)^d$ we shall consider the norm

$$|||u|||_\infty = \max \{ \|u_1\|_\infty, \dots, \|u_d\|_\infty \}, \quad u \in \mathcal{BC}(\mathbb{R}^n)^d.$$

Since $\mathcal{B}_{k'}^\infty(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$ there is $C \geq 1$ so that

$$\|\cdot\|_\infty \leq C \|\cdot\|_{k',\infty}$$

According to Corollary 4.8 $\mathcal{BC}^\infty(\mathbb{R}^n)$ is dense in $\mathcal{B}_{k(k')}^\infty(\mathbb{R}^n)$. Therefore we find $v = (v_1, \dots, v_d) \in \mathcal{BC}^\infty(\mathbb{R}^n)^d$ so that

$$|||u - v|||_{k',\infty} < r/C.$$

Then

$$|||u - v|||_\infty \leq C |||u - v|||_{k',\infty} < r.$$

Using the last estimate we show that $\overline{v(\mathbb{R}^n)} \subset \bigcup_{x \in \mathbb{R}^n} B(u(x); r)$. Indeed, if $z \in \overline{v(\mathbb{R}^n)}$, then there is $x \in \mathbb{R}^n$ such that

$$|z - v(x)|_\infty < r - |||v - u|||_\infty$$

It follows that

$$\begin{aligned} |z - u(x)|_\infty &\leq |z - v(x)|_\infty + |v(x) - u(x)|_\infty \\ &\leq |z - v(x)|_\infty + |||v - u|||_\infty \\ &< r - |||v - u|||_\infty + |||v - u|||_\infty = r \end{aligned}$$

so $z \in B(u(x); r)$.

From $\overline{v(\mathbb{R}^n)} \subset \bigcup_{x \in \mathbb{R}^n} B(u(x); r)$ we get

$$\overline{v(\mathbb{R}^n)} + \overline{B(0; 3r)} \subset \bigcup_{x \in \mathbb{R}^n} B(u(x); 4r) \subset \Omega,$$

hence the map

$$\mathbb{R}^n \times \overline{B(0; 3r)} \ni (x, \zeta) \rightarrow \Phi(v(x) + \zeta) \in \mathbb{C}.$$

is well defined. Let $\Gamma(r)$ denote the polydisc $(\partial \mathbb{D}(0, 3r))^d$. Since $\overline{v(\mathbb{R}^n)} + \Gamma(r) \subset \Omega$ is a compact subset, the map

$$\Gamma(r) \ni \zeta \rightarrow \Phi(\zeta + v) \in \mathcal{BC}^{m_{k'}}(\mathbb{R}^n) \subset \mathcal{B}_{k'}^\infty(\mathbb{R}^n)$$

is continuous.

On the other hand we have

$$(\zeta_1 + v_1 - u_1)^{-1}, \dots, (\zeta_d + v_d - u_d)^{-1} \in \mathcal{B}_{k'}^\infty(\mathbb{R}^n)$$

because $\|u_1 - v_1\|_{k',\infty}, \dots, \|u_d - v_d\|_{k',\infty} < r/C \leq r$ and $|\zeta_1| = \dots = |\zeta_d| = 3r$.

It follows that the integral

$$(4.1) \quad h = \frac{1}{(2\pi i)^d} \int_{\Gamma(r)} \frac{\Phi(\zeta + v)}{(\zeta_1 + v_1 - u_1) \dots (\zeta_d + v_d - u_d)} d\zeta$$

defines an element $h \in \mathcal{B}_{k'}^\infty(\mathbb{R}^n)$.

Let

$$\delta_x : \mathcal{B}_{k'}^\infty(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n) \rightarrow \mathbb{C}, \quad w \rightarrow w(x),$$

be the evaluation functional at $x \in \mathbb{R}^n$. Then

$$\begin{aligned} h(x) &= \frac{1}{(2\pi i)^d} \int_{\Gamma(r)} \frac{\Phi(\zeta + v(x))}{(\zeta_1 - (u_1(x) - v_1(x))) \dots (\zeta_d - (u_d(x) - v_d(x)))} d\zeta \\ &= \Phi(\zeta + v(x))|_{\zeta=u(x)-v(x)} = \Phi(u(x)) \end{aligned}$$

because $|u(x) - v(x)|_\infty \leq \|u - v\|_\infty < r$, so $u(x) - v(x)$ is within polydisc $\Gamma(r)$. Hence $h = \Phi \circ u \equiv \Phi(u) \in \mathcal{B}_{k'}^\infty(\mathbb{R}^n)$.

(b) Let $\varepsilon_0 \in (0, 1]$ be such that for any $0 < \varepsilon \leq \varepsilon_0$ we have

$$\|u - u_\varepsilon\|_{k', \infty} < r/C.$$

Then $\|u - u_\varepsilon\|_\infty \leq C\|u - u_\varepsilon\|_{k', \infty} < r$ and $\overline{u_\varepsilon(\mathbb{R}^n)} \subset \bigcup_{x \in \mathbb{R}^n} B(u(x); r) \subset \Omega$ for every $0 < \varepsilon \leq \varepsilon_0$.

On the other hand we have $\|v - u_\varepsilon\|_{k', \infty} \leq \|v - u\|_{k', \infty} + \|u - u_\varepsilon\|_{k', \infty} < r/C + r/C \leq 2r$. It follows that

$$(\zeta_1 + v_1 - u_{\varepsilon 1})^{-1}, \dots, (\zeta_d + v_d - u_{\varepsilon d})^{-1} \in \mathcal{B}_{k'}^\infty(\mathbb{R}^n)$$

because $\|u_{\varepsilon 1} - v_1\|_{k', \infty}, \dots, \|u_{\varepsilon d} - v_d\|_{k', \infty} < 2r$ and $|\zeta_1| = \dots = |\zeta_d| = 3r$.

We obtain that

$$\begin{aligned} \Phi(u_\varepsilon) &= \frac{1}{(2\pi i)^d} \int_{\Gamma(r)} \frac{\Phi(\zeta + v)}{(\zeta_1 + v_1 - u_{\varepsilon 1}) \dots (\zeta_d + v_d - u_{\varepsilon d})} d\zeta \\ &\rightarrow \frac{1}{(2\pi i)^d} \int_{\Gamma(r)} \frac{\Phi(\zeta + v)}{(\zeta_1 + v_1 - u_1) \dots (\zeta_d + v_d - u_d)} d\zeta = \Phi(u) \end{aligned}$$

as $\varepsilon \rightarrow 0$. □

Remark 4.10. According to Coquand and Stolzenberg [CS], this type of representation formula, (4.1), was introduced more than 60 years ago by A. P. Calderón.

5. THE SPACES \mathcal{B}_k^p AND SCHATTEN-VON NEUMANN CLASS PROPERTIES FOR PSEUDO-DIFFERENTIAL OPERATORS

We begin this section with some interpolation results of \mathcal{B}_k^p spaces.

For $k \in \mathcal{K}(\mathbb{R}^n)$, if

$$\begin{aligned} F_k &= \{v \in \mathcal{S}'(\mathbb{R}^n) : kv \in L^2(\mathbb{R}^n)\}, \\ \|v\|_{F_k} &= \|kv\|_{L^2}, \quad v \in F_k, \end{aligned}$$

then the Fourier transform \mathcal{F} is an isometry (up to a constant factor) from $\mathcal{B}_k(\mathbb{R}^n)$ onto F_k and the inverse Fourier transform \mathcal{F}^{-1} is an isometry (up to a constant factor) from F_k onto $\mathcal{B}_k(\mathbb{R}^n)$. The interpolation property implies then that \mathcal{F} maps continuously $[\mathcal{B}_{k_0}(\mathbb{R}^n), \mathcal{B}_{k_1}(\mathbb{R}^n)]_\theta$ into $[F_{k_0}, F_{k_1}]_\theta$ and \mathcal{F}^{-1} maps continuously $[F_{k_0}, F_{k_1}]_\theta$ into $[\mathcal{B}_{k_0}(\mathbb{R}^n), \mathcal{B}_{k_1}(\mathbb{R}^n)]_\theta$, so that $[\mathcal{B}_{k_0}(\mathbb{R}^n), \mathcal{B}_{k_1}(\mathbb{R}^n)]_\theta$ coincides with the tempered distributions whose Fourier transform belongs to $[F_{k_0}, F_{k_1}]_\theta$ (and one deduces in the same way that it is an isometry if one uses the corresponding norms). Identifying interpolation spaces between spaces $\mathcal{B}_k(\mathbb{R}^n)$ is then the same question as interpolating between some L^2 spaces with weights. The following lemma is a consequence of this remark and Theorem 1.18.5 in [Tri].

Lemma 5.1. If $k_0, k_1 \in \mathcal{K}(\mathbb{R}^n)$, $0 < \theta < 1$ and $k = k_0^{1-\theta} \cdot k_1^\theta \in \mathcal{K}(\mathbb{R}^n)$, then

$$[\mathcal{B}_{k_0}(\mathbb{R}^n), \mathcal{B}_{k_1}(\mathbb{R}^n)]_\theta = \mathcal{B}_k(\mathbb{R}^n).$$

Using the results of [Tri] Subsection 1.18.1 we obtain the following corollary.

Corollary 5.2. *Let $k_0, k_1 \in \mathcal{K}(\mathbb{R}^n)$, $1 \leq p_0 < \infty$, $1 \leq p_1 \leq \infty$, $0 < \theta < 1$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad k = k_0^{1-\theta} \cdot k_1^\theta \in \mathcal{K}(\mathbb{R}^n).$$

Then

$$[l^{p_0}(\mathbb{Z}^n, \mathcal{B}_{k_0}(\mathbb{R}^n)), l^{p_1}(\mathbb{Z}^n, \mathcal{B}_{k_1}(\mathbb{R}^n))]_\theta = l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n)).$$

We pass now to the Kato-Hörmander spaces \mathcal{B}_k^p . We choose $\chi_{\mathbb{Z}^n} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ so that $\sum_{\gamma \in \mathbb{Z}^n} \chi_{\mathbb{Z}^n}(\cdot - \gamma) = 1$. For $\gamma \in \mathbb{Z}^n$ we define the operator

$$S_\gamma : \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n), \quad S_\gamma u = (\tau_\gamma \chi_{\mathbb{Z}^n}) u.$$

Now from the definition of \mathcal{B}_k^p it follows that the linear operator

$$S : \mathcal{B}_k^p(\mathbb{R}^n) \rightarrow l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n)), \quad Su = (S_\gamma u)_{\gamma \in \mathbb{Z}^n}$$

is well defined and continuous.

On the other hand, for any $\chi \in \mathcal{C}_0^\infty$ the operator

$$\begin{aligned} R_\chi : l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n)) &\rightarrow \mathcal{B}_k^p(\mathbb{R}^n), \\ R_\chi \left((u_\gamma)_{\gamma \in \mathbb{Z}^n} \right) &= \sum_{\gamma \in \mathbb{Z}^n} (\tau_\gamma \chi) u_\gamma \end{aligned}$$

is well defined and continuous.

Let $\underline{u} = (u_\gamma)_{\gamma \in \mathbb{Z}^n} \in l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n))$. Using Proposition 2.7 we get

$$\|(\tau_{\gamma'} \chi_{\mathbb{Z}^n}) (\tau_\gamma \chi) u_\gamma\|_{\mathcal{B}_k} \leq Cst \sup_{|\alpha+\beta| \leq m_k} |((\tau_{\gamma'} \partial^\alpha \chi_{\mathbb{Z}^n}) (\tau_\gamma \partial^\beta \chi))| \|u_\gamma\|_{\mathcal{B}_k}.$$

where $m_k = [N + \frac{n+1}{2}] + 1$. Now for some continuous seminorm $p = p_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} |((\tau_{\gamma'} \partial^\alpha \chi_{\mathbb{Z}^n}) (\tau_\gamma \partial^\beta \chi)) (x)| &\leq p(\chi_{\mathbb{Z}^n}) p(\chi) \langle x - \gamma' \rangle^{-2(n+1)} \langle x - \gamma \rangle^{-2(n+1)} \\ &\leq 2^{n+1} p(\chi_{\mathbb{Z}^n}) p(\chi) \langle 2x - \gamma' - \gamma \rangle^{-n-1} \langle \gamma' - \gamma \rangle^{-n-1} \\ &\leq 2^{n+1} p(\chi_{\mathbb{Z}^n}) p(\chi) \langle \gamma' - \gamma \rangle^{-n-1}, \quad |\alpha + \beta| \leq m_k. \end{aligned}$$

Hence

$$\begin{aligned} \sup_{|\alpha+\beta| \leq m_k} |((\tau_{\gamma'} \partial^\alpha \chi_{\mathbb{Z}^n}) (\tau_\gamma \partial^\beta \chi))| &\leq 2^{n+1} p(\chi_{\mathbb{Z}^n}) p(\chi) \langle \gamma' - \gamma \rangle^{-n-1}, \\ \|(\tau_{\gamma'} \chi_{\mathbb{Z}^n}) (\tau_\gamma \chi) u_\gamma\|_{\mathcal{B}_k} &\leq C(n, k, \chi_{\mathbb{Z}^n}, \chi) \langle \gamma' - \gamma \rangle^{-n-1} \|u_\gamma\|_{\mathcal{B}_k}. \end{aligned}$$

The last estimate implies that

$$\|(\tau_{\gamma'} \chi_{\mathbb{Z}^n}) R_\chi(\underline{u})\|_{\mathcal{B}_k} \leq C(n, k, \chi_{\mathbb{Z}^n}, \chi) \sum_{\gamma \in \mathbb{Z}^n} \langle \gamma' - \gamma \rangle^{-n-1} \|u_\gamma\|_{\mathcal{B}_k}.$$

Now Schur's lemma implies the result

$$\left(\sum_{\gamma' \in \mathbb{Z}^n} \|(\tau_{\gamma'} \chi_{\mathbb{Z}^n}) R_\chi(\underline{u})\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}} \leq C'(n, k, \chi_{\mathbb{Z}^n}, \chi) \left\| \langle \cdot \rangle^{-n-1} \right\|_{L^1} \left(\sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^p \right)^{\frac{1}{p}}.$$

If $\chi = 1$ on a neighborhood of $\text{supp} \chi_{\mathbb{Z}^n}$, then $\chi \chi_{\mathbb{Z}^n} = \chi_{\mathbb{Z}^n}$ and as a consequence $R_\chi S = \text{Id}_{\mathcal{B}_k^p(\mathbb{R}^n)}$:

$$\begin{aligned} R_\chi S u &= \sum_{\gamma \in \mathbb{Z}^n} (\tau_\gamma \chi) S_\gamma u = \sum_{\gamma \in \mathbb{Z}^n} (\tau_\gamma \chi) (\tau_\gamma \chi_{\mathbb{Z}^n}) u \\ &= \sum_{\gamma \in \mathbb{Z}^n} (\tau_\gamma \chi_{\mathbb{Z}^n}) u = u. \end{aligned}$$

Thus we proved the following result.

Proposition 5.3. *Under the above conditions, the operator $R_\chi : l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n)) \rightarrow \mathcal{B}_k^p(\mathbb{R}^n)$ is a retraction and the operator $S : \mathcal{B}_k^p(\mathbb{R}^n) \rightarrow l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n))$ is a coretraction.*

Corollary 5.4. *Let $k_0, k_1 \in \mathcal{K}(\mathbb{R}^n)$, $1 \leq p_0 < \infty$, $1 \leq p_1 \leq \infty$, $0 < \theta < 1$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad k = k_0^{1-\theta} \cdot k_1^\theta \in \mathcal{K}(\mathbb{R}^n).$$

Then

$$[\mathcal{B}_{k_0}^{p_0}(\mathbb{R}^n), \mathcal{B}_{k_1}^{p_1}(\mathbb{R}^n)]_\theta = \mathcal{B}_k^p(\mathbb{R}^n).$$

Proof. The last part of Theorem 3.6 (d) shows that $\{\mathcal{B}_{k_0}^{p_0}(\mathbb{R}^n), \mathcal{B}_{k_1}^{p_1}(\mathbb{R}^n)\}$ is an interpolation couple (in the sense of the notations of Subsection 1.2.1 of [Tri] one can choose $\mathcal{A} = \mathcal{S}'(\mathbb{R}^n)$). If F is an interpolation functor, then one obtains by Theorem 1.2.4 of [Tri] that

$$\|u\|_{F(\{\mathcal{B}_{k_0}^{p_0}(\mathbb{R}^n), \mathcal{B}_{k_1}^{p_1}(\mathbb{R}^n)\})} \approx \left\| (S_\gamma u)_{\gamma \in \mathbb{Z}^n} \right\|_{F(\{l^{p_0}(\mathbb{Z}^n, \mathcal{B}_{k_0}), l^{p_1}(\mathbb{Z}^n, \mathcal{B}_{k_1})\})}$$

By specialization we obtain

$$\begin{aligned} \|u\|_{[\mathcal{B}_{k_0}^{p_0}(\mathbb{R}^n), \mathcal{B}_{k_1}^{p_1}(\mathbb{R}^n)]_\theta} &\approx \left\| (S_\gamma u)_{\gamma \in \mathbb{Z}^n} \right\|_{[l^{p_0}(\mathbb{Z}^n, \mathcal{B}_{k_0}), l^{p_1}(\mathbb{Z}^n, \mathcal{B}_{k_1})]_\theta} \\ &\approx \left\| (S_\gamma u)_{\gamma \in \mathbb{Z}^n} \right\|_{l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n))} \\ &\approx \|u\|_{\mathcal{B}_k^p(\mathbb{R}^n)} \end{aligned}$$

□

In addition to the above interpolation results we need an embedding theorem which we shall prove below. First we shall recall the definition of spaces that appear in this theorem.

Definition 5.5. *Let $1 \leq p \leq \infty$. We say that a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $S_w^p(\mathbb{R}^n)$ if there is $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ such that the measurable function*

$$\begin{aligned} U_{\chi, p} : \mathbb{R}^n &\rightarrow [0, +\infty), \\ U_{\chi, p}(\xi) &= \begin{cases} \sup_{y \in \mathbb{R}^n} |\widehat{u \tau_y \chi}(\xi)| & \text{if } p = \infty \\ \left(\int |\widehat{u \tau_y \chi}(\xi)|^p dy \right)^{1/p} & \text{if } 1 \leq p < \infty \end{cases}, \\ \widehat{u \tau_y \chi}(\xi) &= \left\langle u, e^{-i(\cdot, \xi)} \chi(\cdot - y) \right\rangle. \end{aligned}$$

belongs to $L^1(\mathbb{R}^n)$.

These spaces are special cases of modulation spaces which were introduced by Hans Georg Feichtinger in 1983. They were used by many authors (Boulkhemair, Gröchenig, Heil, Sjöstrand, Toft ...) in the analysis of pseudo-differential operators defined by symbols more general than usual.

Now we give some properties of these spaces.

Proposition 5.6. (a) *Let $u \in S_w^p(\mathbb{R}^n)$ and let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Then the measurable function*

$$U_{\chi,p} : \mathbb{R}^n \rightarrow [0, +\infty),$$

$$U_{\chi,p}(\xi) = \begin{cases} \sup_{y \in \mathbb{R}^n} |\widehat{u\tau_y\chi}(\xi)| & \text{if } p = \infty \\ \left(\int |\widehat{u\tau_y\chi}(\xi)|^p dy \right)^{1/p} & \text{if } 1 \leq p < \infty \end{cases},$$

$$\widehat{u\tau_y\chi}(\xi) = \left\langle u, e^{-i(\cdot, \xi)} \chi(\cdot - y) \right\rangle.$$

belongs to $L^1(\mathbb{R}^n)$.

(b) *If we fix $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ and if we put*

$$\|u\|_{S_{w,\chi}^p} = \int U_{\chi,p}(\xi) d\xi = \|U_{\chi,p}\|_{L^1}, \quad u \in S_w(\mathbb{R}^n),$$

then $\|\cdot\|_{S_{w,\chi}^p}$ is a norm on $S_w^p(\mathbb{R}^n)$ and the topology that defines does not depend on the choice of the function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$.

(c) *Let $1 \leq p \leq q \leq \infty$. Then*

$$S_w^1(\mathbb{R}^n) \subset S_w^p(\mathbb{R}^n) \subset S_w^q(\mathbb{R}^n) \subset S_w^\infty(\mathbb{R}^n) = S_w(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

(d) *If $\lambda \in \mathbf{End}_{\mathbb{R}}(\mathbb{R}^n)$ is invertible and $u \in S_w^p(\mathbb{R}^n)$, then $u_\lambda = u \circ \lambda \in S_w^p(\mathbb{R}^n)$ and there is $C \in (0, +\infty)$ independent of u and λ such that*

$$\|u_\lambda\|_{S_w^p} \leq C |\det \lambda|^{-n/p} (1 + \|\lambda\|)^n \|u\|_{S_w^p}.$$

A proof of this proposition can be found for instance in [A1].

Lemma 5.7. *Let $k \in \mathcal{K}(\mathbb{R}^n)$ and $1 \leq p \leq \infty$. If $1/k \in L^1(\mathbb{R}^n)$, then $\mathcal{B}_k^p(\mathbb{R}^n) \hookrightarrow S_w^p(\mathbb{R}^n)$.*

Proof. Let $u \in \mathcal{B}_k^p(\mathbb{R}^n)$. Let $\chi, \tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ be such that $\tilde{\chi} = 1$ on $\text{supp } \chi$. For $y \in \mathbb{R}^n$ we have

$$u\tau_y\chi = (u\tau_y\tilde{\chi})(\tau_y\chi) \Rightarrow \widehat{u\tau_y\chi} = (2\pi)^{-n} \widehat{u\tau_y\tilde{\chi}} * \widehat{\tau_y\chi}.$$

Multiplying by $k(\xi)$ and noting the inequality $k(\xi) \leq M_k(\xi - \eta)k(\eta)$, we obtain

$$\begin{aligned} k(\xi) |\widehat{u\tau_y\chi}(\xi)| &\leq (2\pi)^{-n} \int k(\eta) |\widehat{u\tau_y\tilde{\chi}}(\eta)| M_k(\xi - \eta) |\widehat{\tau_y\chi}(\xi - \eta)| d\xi \\ &\leq (2\pi)^{-n} \left\| k u \tau_y \tilde{\chi} \right\|_{L^2} \|M_k \widehat{\tau_y\chi}\|_{L^2} \\ &= (2\pi)^{-n} \|u \tau_y \tilde{\chi}\|_{\mathcal{B}_k} \|\chi\|_{\mathcal{B}_{M_k}}, \end{aligned}$$

hence

$$|\widehat{u\tau_y\chi}(\xi)| \leq (2\pi)^{-n} \|u \tau_y \tilde{\chi}\|_{\mathcal{B}_k} \|\chi\|_{\mathcal{B}_{M_k}} \frac{1}{k(\xi)}.$$

It follows that

$$\begin{aligned}
U_{\chi,p}(\xi) &\leq (2\pi)^{-n} \|\chi\|_{\mathcal{B}_{M_k}} \frac{1}{k(\xi)} \begin{cases} \sup_{y \in \mathbb{R}^n} \|u\tau_y \tilde{\chi}\|_{\mathcal{B}_k} & \text{if } p = \infty \\ \left(\int \|u\tau_y \tilde{\chi}\|_{\mathcal{B}_k}^p dy \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \end{cases} \\
&\leq (2\pi)^{-n} \|\chi\|_{\mathcal{B}_{M_k}} \frac{1}{k(\xi)} \begin{cases} \|u\|_{k,\infty,\tilde{\chi}} & \text{if } p = \infty \\ \|u\|_{k,p,\tilde{\chi}} & \text{if } 1 \leq p < \infty \end{cases} \\
&\leq (2\pi)^{-n} \|\chi\|_{\mathcal{B}_{M_k}} \|u\|_{k,p,\tilde{\chi}} \frac{1}{k(\xi)},
\end{aligned}$$

which implies that

$$\|u\|_{S_w^p,\chi} = \|U_{\chi,p}\|_{L^1} \leq (2\pi)^{-n} \|\chi\|_{\mathcal{B}_k} \|1/k\|_{L^1} \|u\|_{\mathbf{s},p,\tilde{\chi}}, \quad u \in \mathcal{B}_k^p.$$

□

This embedding theorem allows us to deal with Schatten-von Neumann class properties of pseudo-differential operators.

Let $\tau \in \mathbf{End}_{\mathbb{R}}(\mathbb{R}^n) \equiv M_{n \times n}(\mathbb{R})$, $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, $v \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\begin{aligned}
\mathbf{Op}_{\tau}(a)v(x) &= a^{\tau}(X,D)v(x) \\
&= (2\pi)^{-n} \iint \mathbf{e}^{i\langle x-y,\eta \rangle} a((1-\tau)x + \tau y, \eta) v(y) dy d\eta.
\end{aligned}$$

If $u, v \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned}
\langle \mathbf{Op}_{\tau}(a)v, u \rangle &= (2\pi)^{-n} \iiint \mathbf{e}^{i\langle x-y,\eta \rangle} a((1-\tau)x + \tau y, \eta) u(x) v(y) dx dy d\eta \\
&= \langle ((1 \otimes \mathcal{F}^{-1})a) \circ \mathbf{C}_{\tau}, u \otimes v \rangle,
\end{aligned}$$

where

$$\mathbf{C}_{\tau} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n, \quad \mathbf{C}_{\tau}(x, y) = ((1-\tau)x + \tau y, x - y).$$

We can define $\mathbf{Op}_{\tau}(a)$ as an operator in $\mathcal{B}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ for any $a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$\begin{aligned}
\langle \mathbf{Op}_{\tau}(a)v, u \rangle_{\mathcal{S}, \mathcal{S}'} &= \langle \mathcal{K}_{\mathbf{Op}_{\tau}(a)}, u \otimes v \rangle, \\
\mathcal{K}_{\mathbf{Op}_{\tau}(a)} &= ((1 \otimes \mathcal{F}^{-1})a) \circ \mathbf{C}_{\tau}
\end{aligned}$$

Theorem 5.8. *Let $k \in \mathcal{K}(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $1/k \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.*

(a) *Let $1 \leq p < \infty$, $\tau \in \mathbf{End}_{\mathbb{R}}(\mathbb{R}^n) \equiv M_{n \times n}(\mathbb{R})$ and $a \in \mathcal{B}_k^p(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$\mathbf{Op}_{\tau}(a) = a^{\tau}(X, D) \in \mathcal{B}_p(L^2(\mathbb{R}^n)),$$

where $\mathcal{B}_p(L^2(\mathbb{R}^n))$ denote the Schatten ideal of compact operators whose singular values lie in l^p . We have

$$\|\mathbf{Op}_{\tau}(a)\|_{\mathcal{B}_p(L^2(\mathbb{R}^n))} \leq Cst \|a\|_{\mathcal{B}_k^p}.$$

Moreover, the mapping

$$\mathbf{End}_{\mathbb{R}}(\mathbb{R}^n) \ni \tau \rightarrow \mathbf{Op}_{\tau}(a) = a^{\tau}(X, D) \in \mathcal{B}_p(L^2(\mathbb{R}^n))$$

is continuous.

(b) *Let $\tau \in \mathbf{End}_{\mathbb{R}}(\mathbb{R}^n) \equiv M_{n \times n}(\mathbb{R})$ and $a \in \mathcal{B}_k^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$\mathbf{Op}_{\tau}(a) = a^{\tau}(X, D) \in \mathcal{B}(L^2(\mathbb{R}^n)).$$

We have

$$\|\mathbf{Op}_\tau(a)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq Cst \|a\|_{\mathcal{B}_k^\infty}.$$

Moreover, the mapping

$$\mathbf{End}_{\mathbb{R}}(\mathbb{R}^n) \ni \tau \rightarrow \mathbf{Op}_\tau(a) = a^\tau(X, D) \in \mathcal{B}(L^2(\mathbb{R}^n))$$

is continuous.

Proof. This theorem is a consequence of the previous theorem and the fact that it is true for pseudo-differential operators with symbols in $S_w^p(\mathbb{R}^n \times \mathbb{R}^n)$ (see for instance [A1] for $1 \leq p < \infty$ and [B2] for $p = \infty$). \square

Theorem 5.9. *Let $k \in \mathcal{K}(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $1/k \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and $1 \leq p < \infty$. If $\tau \in \mathbf{End}_{\mathbb{R}}(\mathbb{R}^n) \equiv M_{n \times n}(\mathbb{R})$ and $a \in \mathcal{B}_{k|1-2/p|}^p(\mathbb{R}^n \times \mathbb{R}^n)$ then*

$$\mathbf{Op}_\tau(a) = a^\tau(X, D) \in \mathcal{B}_p(L^2(\mathbb{R}^n)).$$

Moreover, the mapping

$$\mathbf{End}_{\mathbb{R}}(\mathbb{R}^n) \ni \tau \rightarrow \mathbf{Op}_\tau(a) = a^\tau(X, D) \in \mathcal{B}_p(L^2(\mathbb{R}^n))$$

is continuous.

Proof. The Schwartz kernel of the operator $\mathbf{Op}_\tau(a)$ is $((1 \otimes \mathcal{F}^{-1})a) \circ \mathbf{C}_\tau$. Therefore, $a \in \mathcal{B}_1^2(\mathbb{R}^n \times \mathbb{R}^n) \equiv L^2(\mathbb{R}^n \times \mathbb{R}^n)$ implies that $\mathbf{Op}_\tau(a) \in \mathcal{B}_2(L^2(\mathbb{R}^n))$. Next we use the interpolation properties of Kato-Hörmander spaces \mathcal{B}_k^p and of the Schatten ideals $\mathcal{B}_p(L^2(\mathbb{R}^n))$ to finish the theorem.

$$\begin{aligned} [\mathcal{B}_1^2(\mathbb{R}^n \times \mathbb{R}^n), \mathcal{B}_k^1(\mathbb{R}^n \times \mathbb{R}^n)]_{\frac{2}{p}-1} &= \mathcal{B}_{k^{2/p-1}}^p(\mathbb{R}^n \times \mathbb{R}^n) \\ [\mathcal{B}_2(L^2(\mathbb{R}^n)), \mathcal{B}_1(L^2(\mathbb{R}^n))]_{\frac{2}{p}-1} &= \mathcal{B}_p(L^2(\mathbb{R}^n)), \quad 1 \leq p \leq 2, \\ [\mathcal{B}_1^2(\mathbb{R}^n \times \mathbb{R}^n), \mathcal{B}_k^\infty(\mathbb{R}^n \times \mathbb{R}^n)]_{1-\frac{2}{p}} &= \mathcal{B}_{k^{1-2/p}}^p(\mathbb{R}^n \times \mathbb{R}^n) \\ [\mathcal{B}_2(L^2(\mathbb{R}^n)), \mathcal{B}(L^2(\mathbb{R}^n))]_{1-\frac{2}{p}} &= \mathcal{B}_p(L^2(\mathbb{R}^n)), \quad 2 \leq p < \infty. \end{aligned}$$

\square

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